



Lecture 25 (Graphs 4)

Minimum Spanning Trees

CS61B, Spring 2024 @ UC Berkeley

Slides credit: Josh Hug

Graph Problem Warmup

Lecture 25, CS61B, Spring 2024

Graph Problem Warmup

Minimum Spanning Trees

- Intro
- The Cut Property

Prim's Algorithm

- Basic Prim's (Demo)
- Optimized Prim's (Demo)
- Prim's Algorithm Code and Runtime

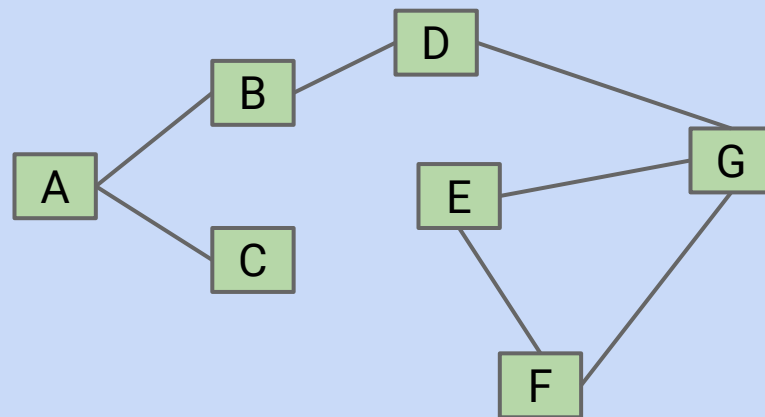
Kruskal's Algorithm:

- Basic Kruskal's (Demo)
- Optimized Kruskal's (Demo)
- Kruskal's vs. Prim's
- Kruskal's Algorithm Code and Runtime

Warm-up Problem

Given an undirected graph, determine if it contains any cycles.

- May use any data structure or algorithm from the course so far.



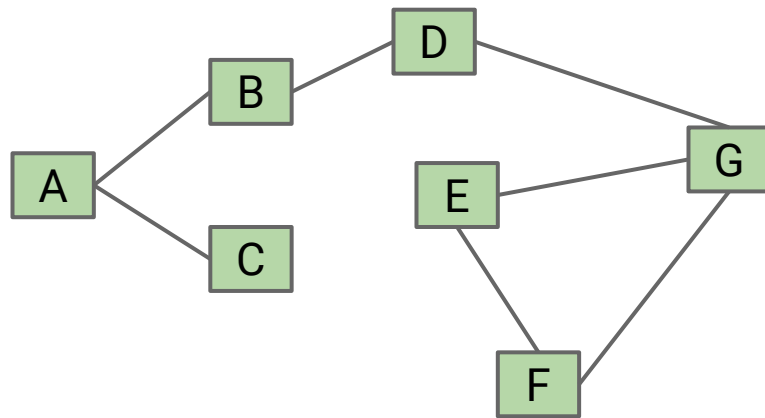
Warm-up Problem

Given an undirected graph, determine if it contains any cycles.

- May use any data structure or algorithm from the course so far.

Approach 1: Do DFS from 0 (arbitrary vertex).

- Keep going until you see a marked vertex.
- Potential danger:
 - 1 looks back at 0 and sees marked.
 - Solution: Just don't count the node you came from.



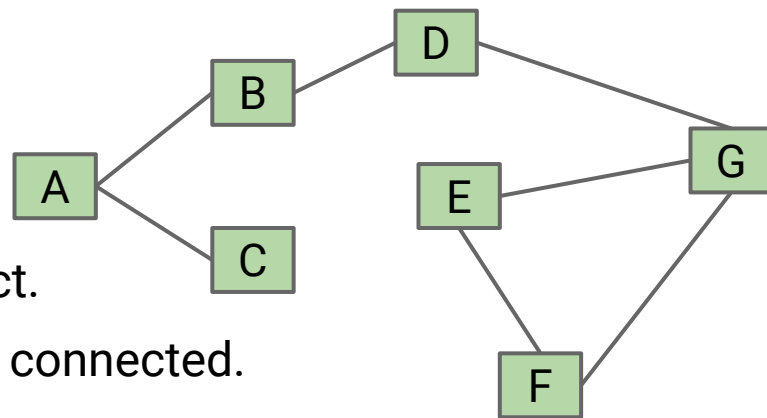
Worst case runtime: $O(V + E)$ -- do study guide problems to reinforce this.

- With some cleverness, can give a tighter bound of $O(V)$ (the number of edges we check is at most V , so $O(V+E) = O(V)$)

Warm-up Problem

Given an undirected graph, determine if it contains any cycles.

- May use any data structure or algorithm from the course so far.



Approach 2: Use a WeightedQuickUnionUF object.

- For each edge, check if the two vertices are connected.
 - If not, union them.
 - If so, there is a cycle.

Worst case runtime: $O(V + E \alpha(V))$ if we have path compression.

- Here $\alpha(V)$ is the [inverse Ackermann function](#) from Disjoint Sets.
- With similar reasoning from before, we can reduce to $O(V \alpha(V))$

Minimum Spanning Trees Intro

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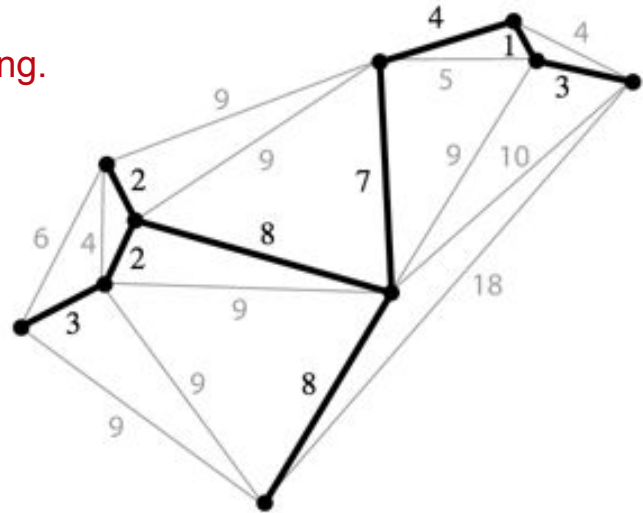
Spanning Trees

Given an **undirected** graph, a **spanning tree** T is a subgraph of G , where T :

- Is connected.
 - Is acyclic.
 - Includes all of the vertices.
- These two properties make it a tree.
- This makes it spanning.

Example:

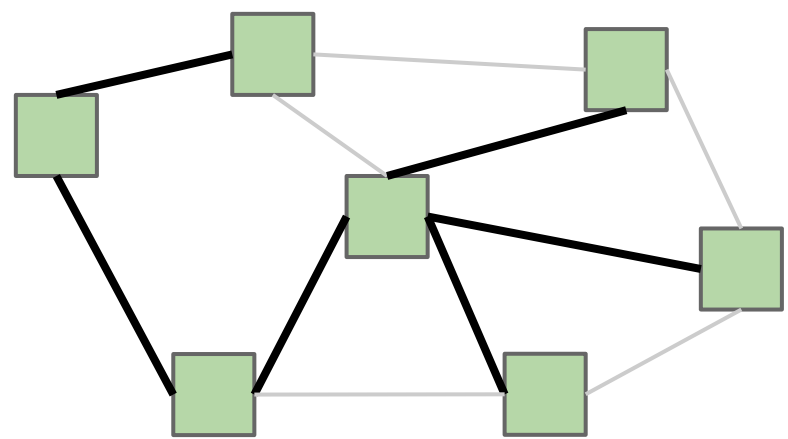
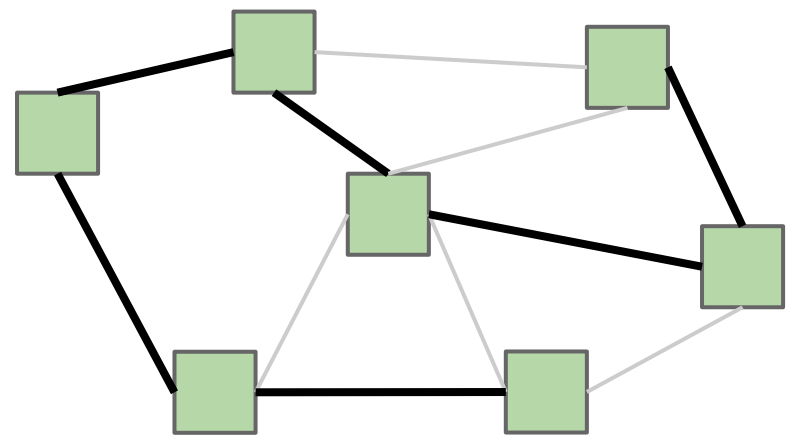
- Spanning tree is the black edges and vertices.



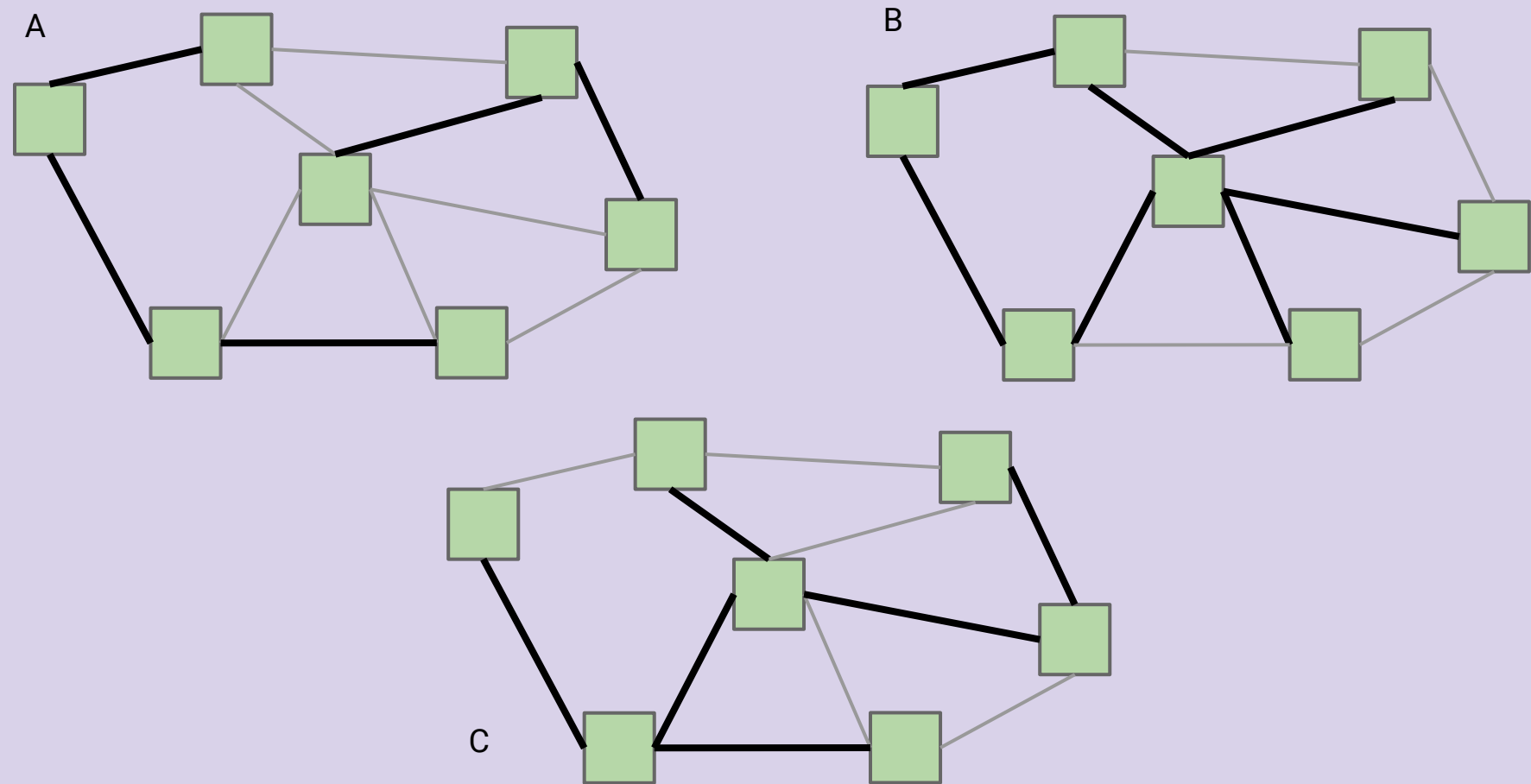
A **minimum spanning tree** is a spanning tree of minimum total weight.

- Example: Network of power lines that connect a bunch of buildings.

Spanning Trees



Which are Spanning Trees? yellkey.com/TODO



Left: Old school handwriting recognition ([link](#))

Right: Medical imaging (e.g. arrangement of nuclei in cancer cells)

For more, see: <http://www.ics.uci.edu/~eppstein/gina/mst.html>

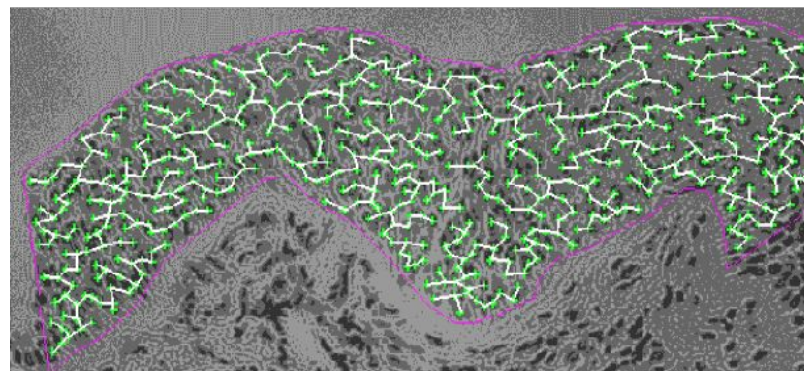
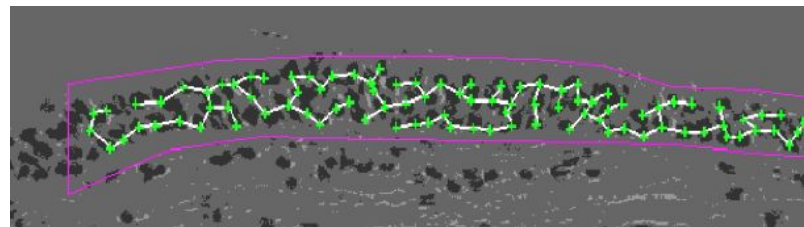
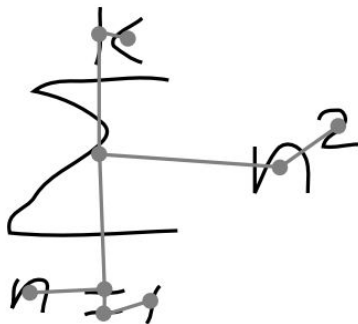


Figure 4-3: A typical minimum spanning tree

These slides are covered in the [web videos](#), but we won't cover them live.

Extra: Minimum Spanning Trees vs. Shortest Paths Trees

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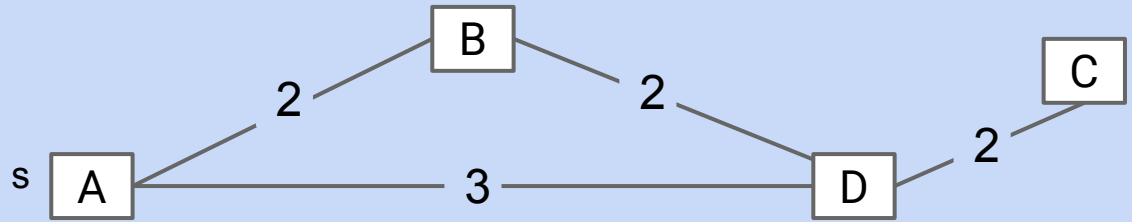
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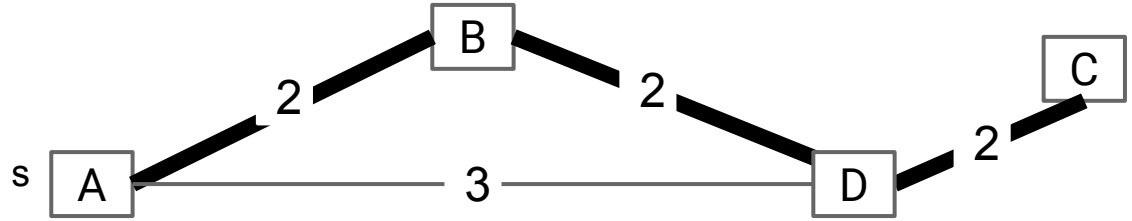
Kruskal's Algorithm:

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Find the MST for the graph.

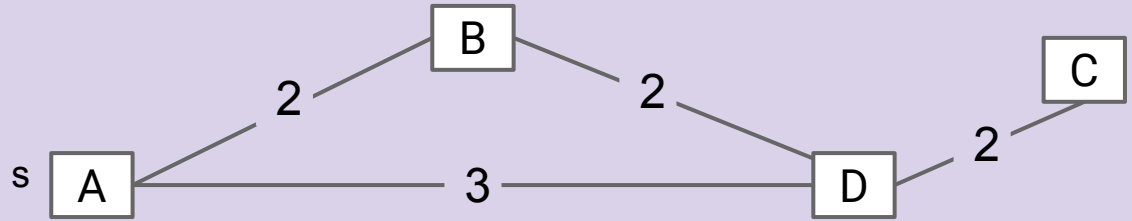


Find the MST for the graph.



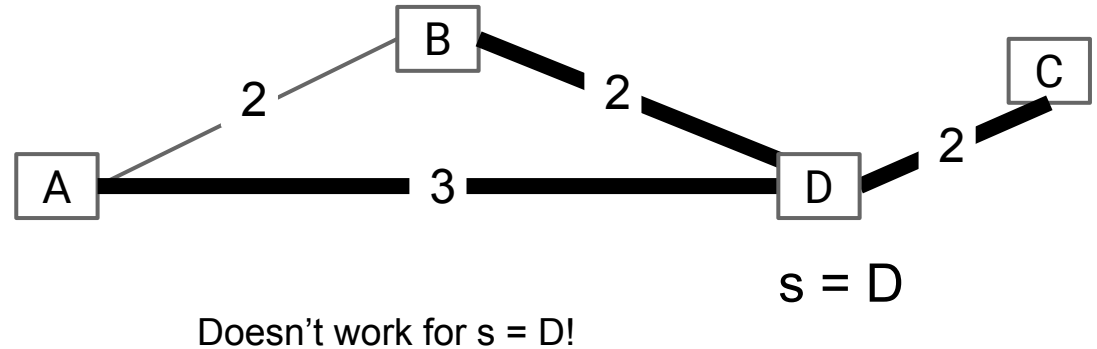
Is the MST for this graph also a shortest paths tree? If so, using which node as the starting node for this SPT?

- A. A
- B. B
- C. C
- D. D
- E. No SPT is an MST.



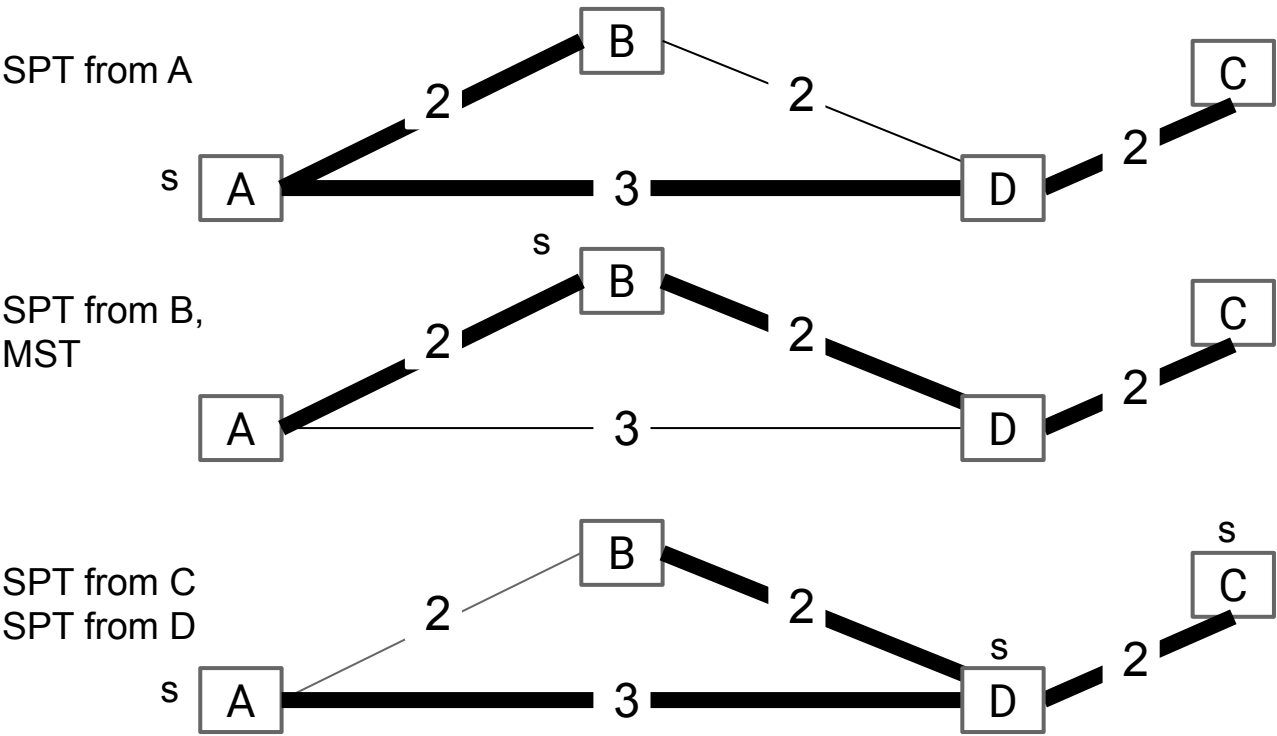
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- A. A
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Is the MST for this graph also a shortest paths tree? If so, using which node as the starting node for this SPT?

- A. A
- B. B**
- C. C
- D. D
- E. No SPT is an MST.



A shortest paths tree depends on the start vertex:

- Because it tells you how to get from a source to EVERYTHING.

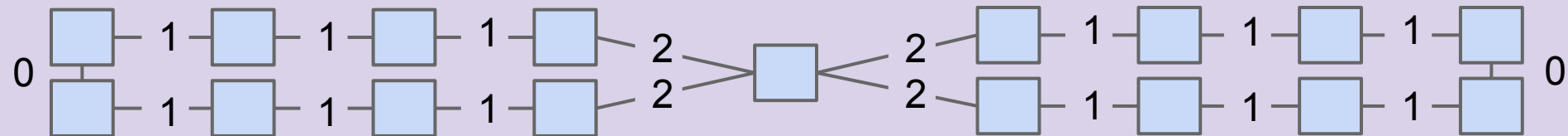
There is no source for a MST.

Nonetheless, the MST sometimes happens to be an SPT for a specific vertex.

Spanning Tree

Give a valid MST for the graph below.

- Hard B level question: Is there a node whose SPT is also the MST?

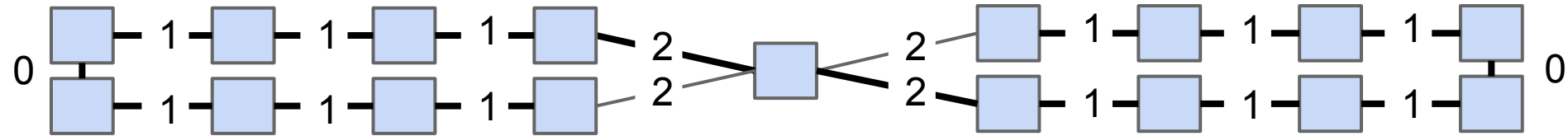


- A. Yes
- B. No

Spanning Tree

Give a valid MST for the graph below.

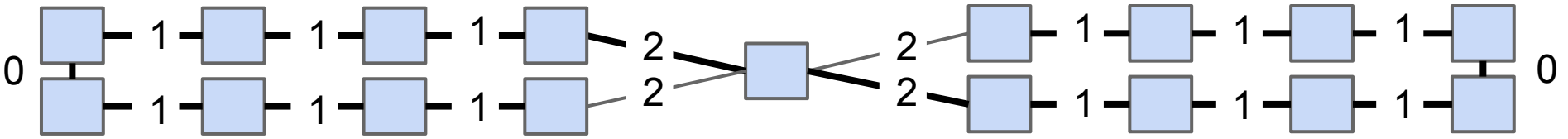
- Is there a node whose SPT is also the MST? [see next slide]



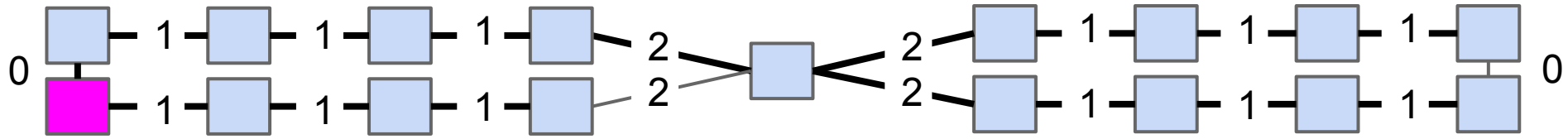
Spanning Tree

Give a valid MST for the graph below.

- Is there a node whose SPT is also the MST?
- **No!** Minimum spanning tree must include only 2 of the 2 weight edges, but the SPT always includes at least 3 of the 2 weight edges.



Example SPT from bottom left vertex:



The Cut Property

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Graph Problem Warmup

Minimum Spanning Trees

- Intro
- **The Cut Property**

Prim's Algorithm

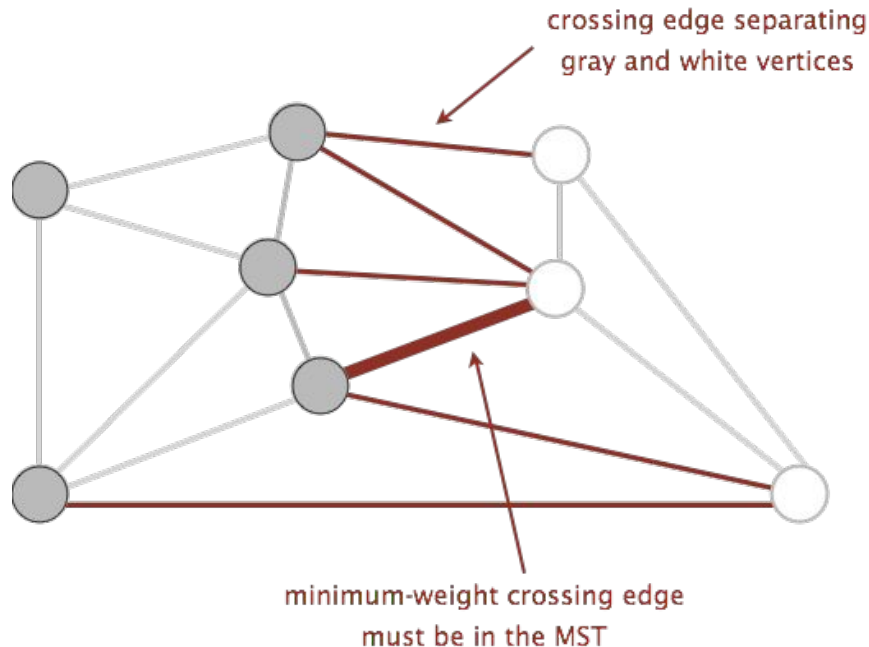
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A Useful Tool for Finding the MST: Cut Property

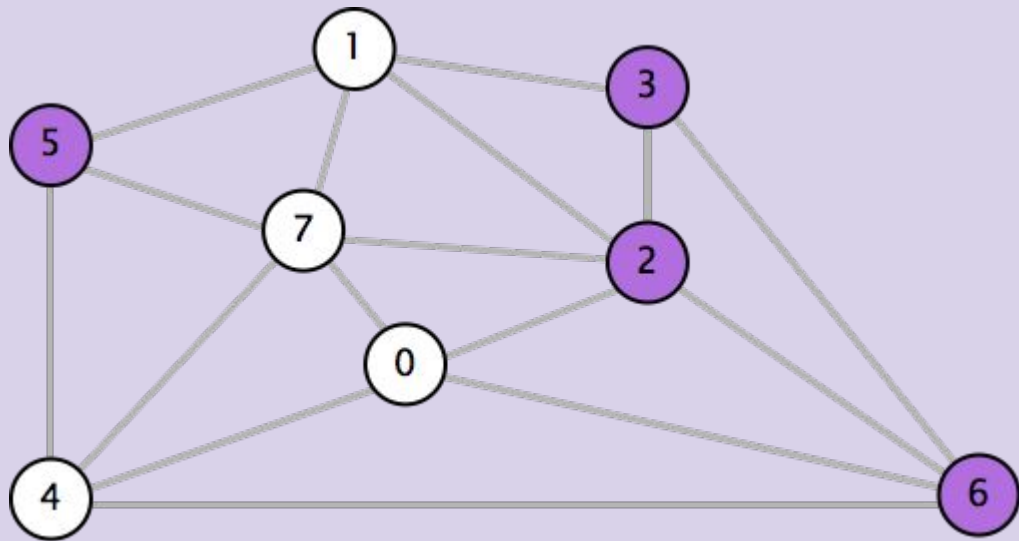
- A **cut** is an assignment of a graph's nodes to two non-empty sets.
- A **crossing edge** is an edge which connects a node from one set to a node from the other set.



Cut property: Given any cut, minimum weight crossing edge is in the MST.

- For rest of today, we'll assume edge weights are unique.

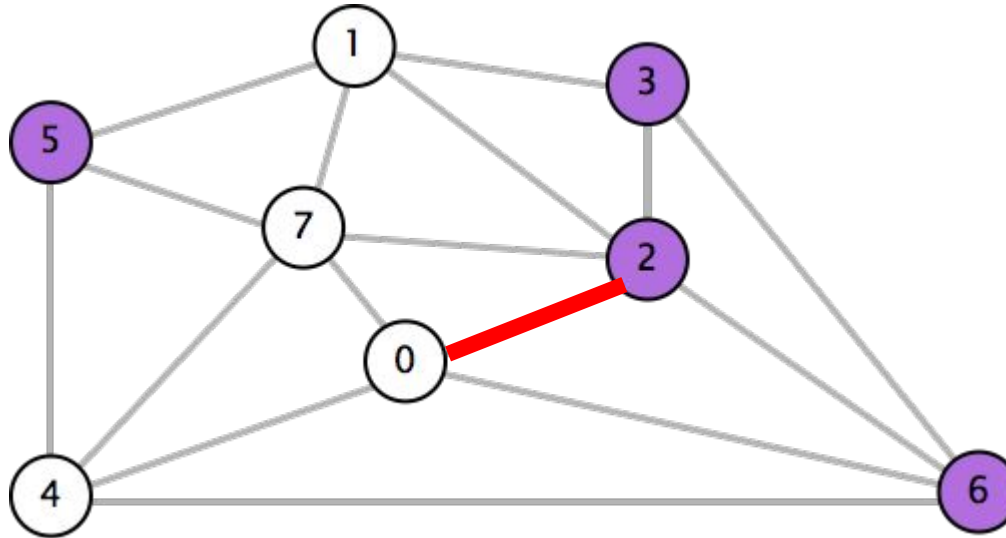
Which edge is the minimum weight edge crossing the cut $\{2, 3, 5, 6\}$?



| | |
|-----|------|
| 0-7 | 0.16 |
| 2-3 | 0.17 |
| 1-7 | 0.19 |
| 0-2 | 0.26 |
| 5-7 | 0.28 |
| 1-3 | 0.29 |
| 1-5 | 0.32 |
| 2-7 | 0.34 |
| 4-5 | 0.35 |
| 1-2 | 0.36 |
| 4-7 | 0.37 |
| 0-4 | 0.38 |
| 6-2 | 0.40 |
| 3-6 | 0.52 |
| 6-0 | 0.58 |
| 6-4 | 0.93 |

Which edge is the minimum weight edge crossing the cut $\{2, 3, 5, 6\}$?

- 0-2. Must be part of the MST!

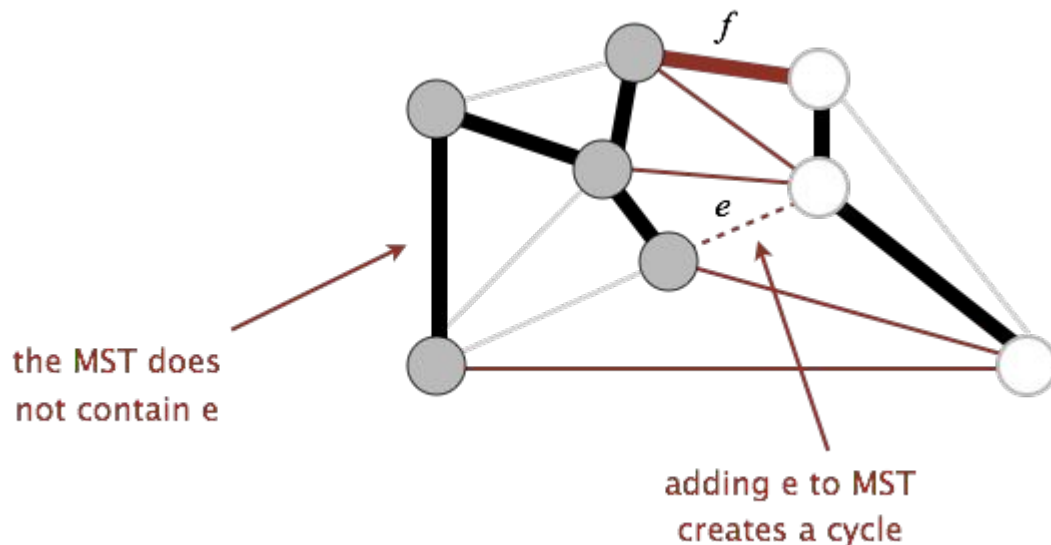


| | |
|------------|-------------|
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Cut Property Proof

Suppose that the minimum crossing edge e were not in the MST.

- Adding e to the MST creates a cycle.
- Some other edge f must also be a crossing edge.
- Removing f and adding e is a lower weight spanning tree.
- Contradiction!



Start with no edges in the MST.

- Find a cut that has no crossing edges in the MST.
- Add smallest crossing edge to the MST.
- Repeat until $V-1$ edges.

This should work, but we need some way of finding a cut with no crossing edges!

- Random isn't a very good idea.

Basic Prim's (Demo)

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Prim's Algorithm

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- Optimized Prim's (Demo)
- Prim's Algorithm Code and Runtime

Kruskal's Algorithm:

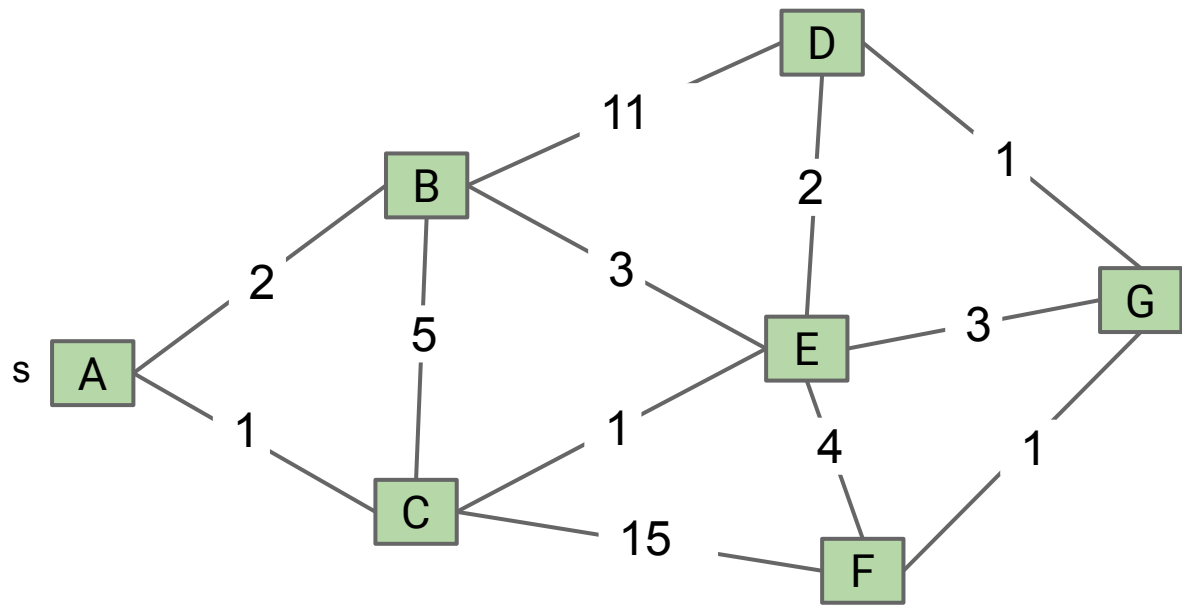
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Prim's Demo (Conceptual)

Start from some arbitrary start node.

- Add shortest edge (mark black) that has one node inside the MST under construction. Repeat until $V-1$ edges.

| Node | edgeTo |
|------|--------|
| A | - |
| B | - |
| C | - |
| D | - |
| E | - |
| F | - |
| G | - |

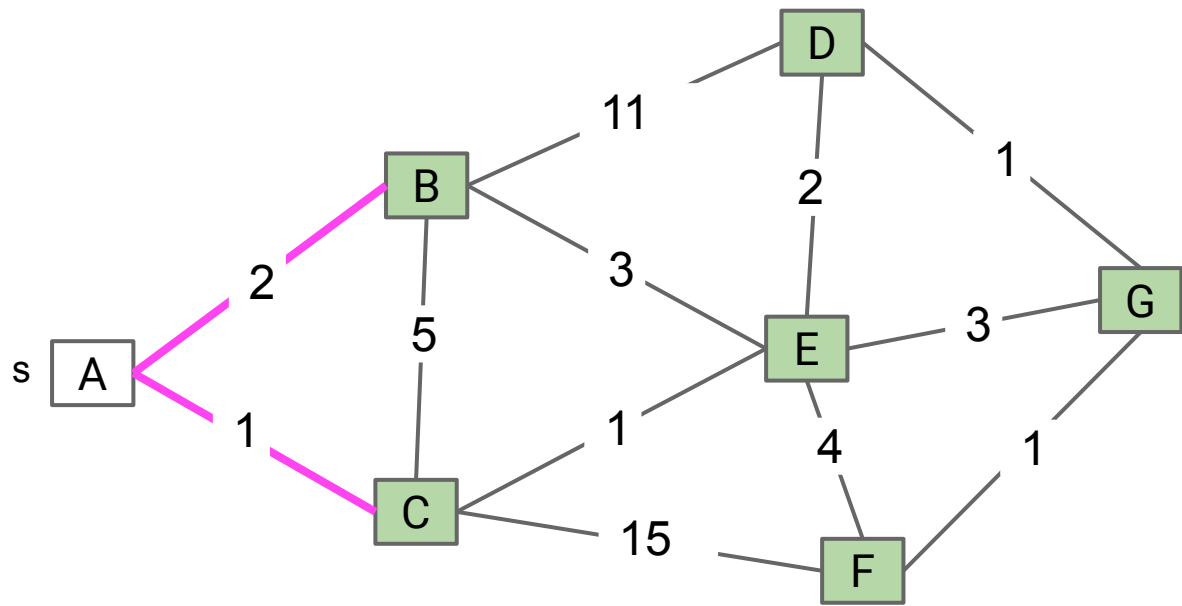


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| G | - |

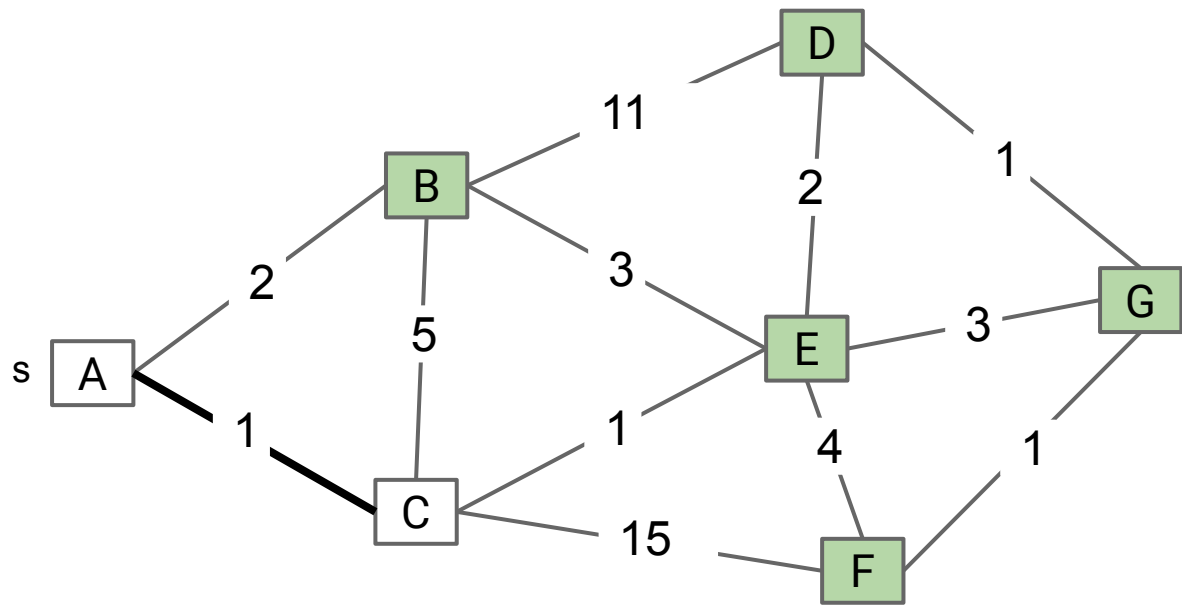


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| C | A |
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| G | - |

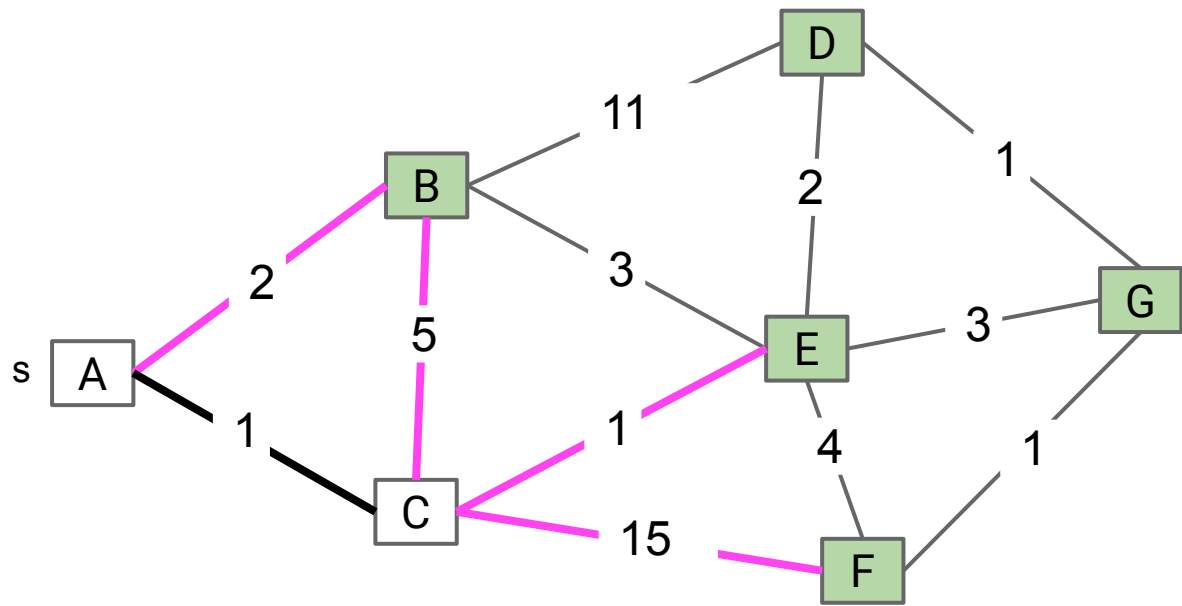


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| F | - |
| G | - |

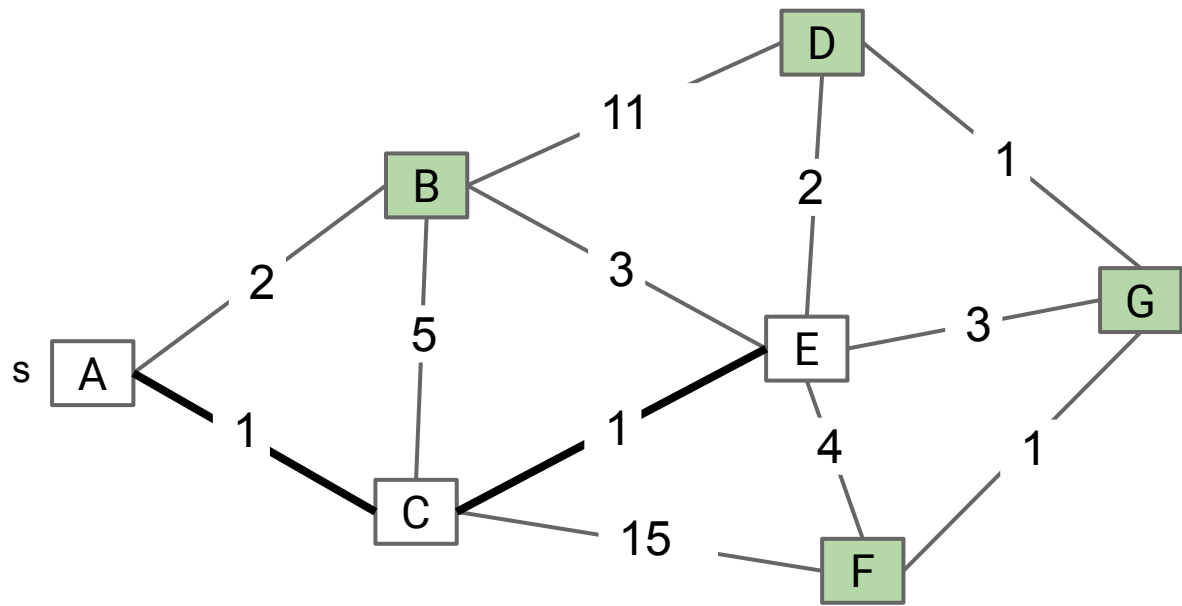


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| E | C |
| F | - |
| G | - |

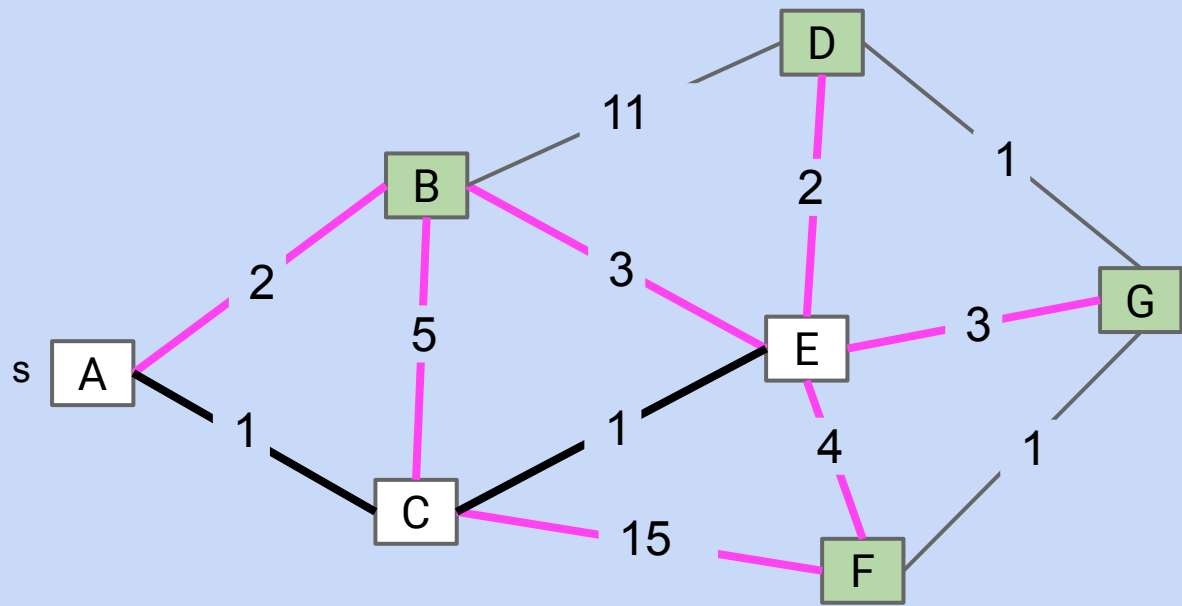


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- Add shortest edge (mark black) that has one node inside the MST under construction. Repeat until $V-1$ edges.

| Node | edgeTo |
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| A | - |
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| C | A |
| D | - |
| E | C |
| F | - |
| G | - |



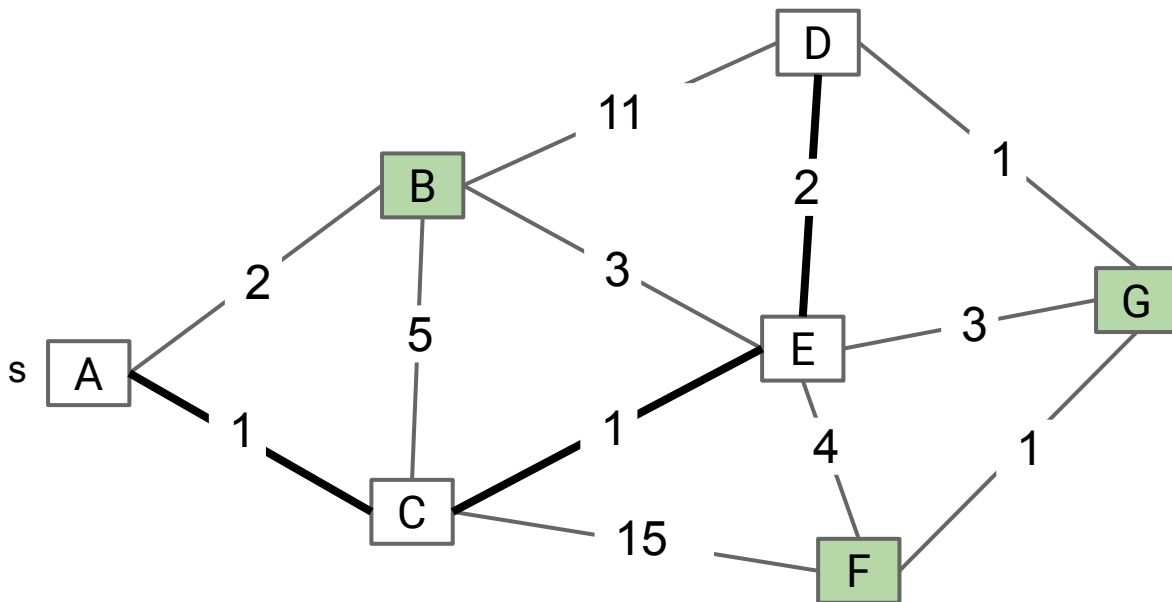
Which edge is added next?

Prim's Demo (Conceptual)

Start from some arbitrary start node.

- Add shortest edge (mark black) that has one node inside the MST under construction. Repeat until $V-1$ edges.

| Node | edgeTo |
|------|--------|
| A | - |
| B | - |
| C | A |
| D | E |
| E | C |
| F | - |
| G | - |



Which edge is added next?

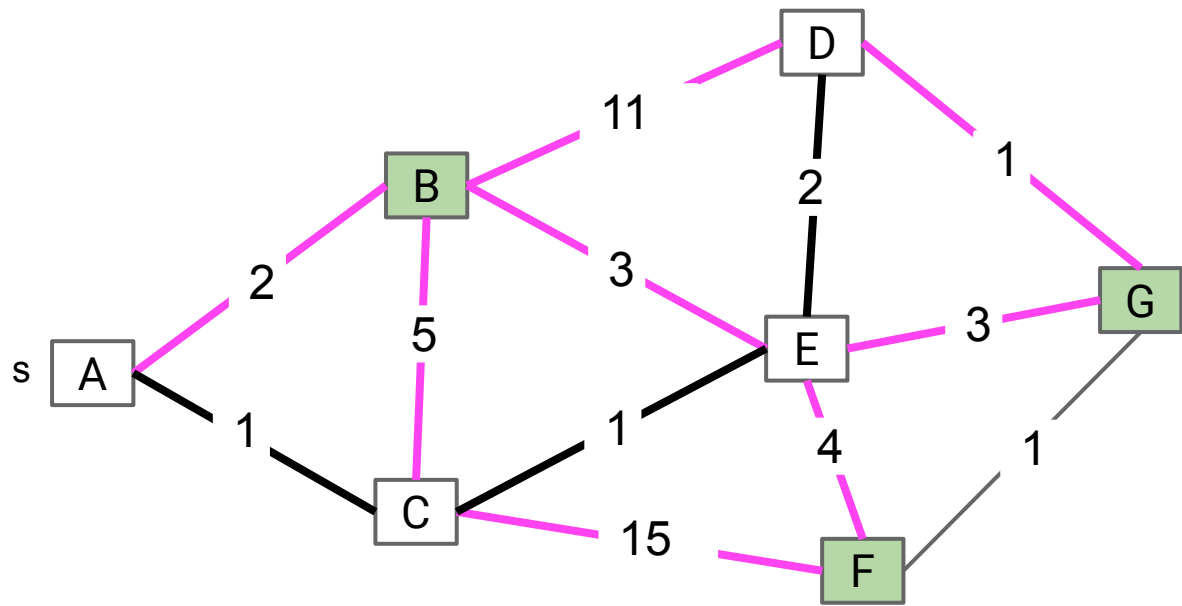
- Either A-B or D-E are guaranteed to work (see exercises for proof)!
- Note: They are not both guaranteed to be in the MST.

Prim's Demo (Conceptual)

Start from some arbitrary start node.

- Add shortest edge (mark black) that has one node inside the MST under construction. Repeat until $V-1$ edges.

| Node | edgeTo |
|------|--------|
| A | - |
| B | - |
| C | A |
| D | E |
| E | C |
| F | - |
| G | - |

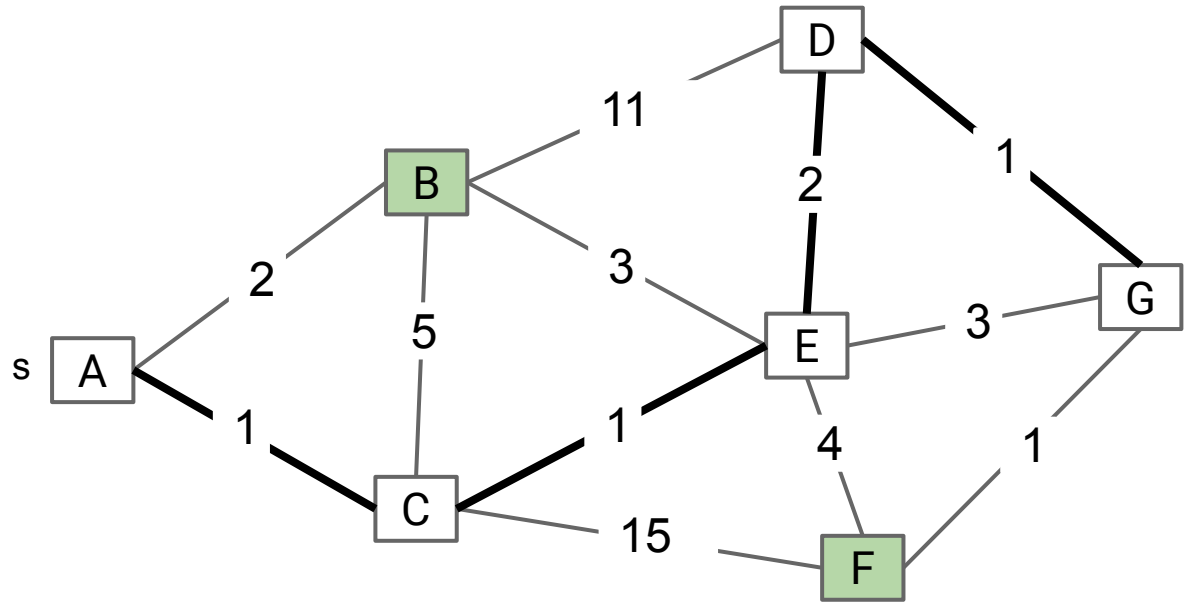


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|------|--------|
| A | - |
| B | - |
| C | A |
| D | E |
| E | C |
| F | - |
| G | D |

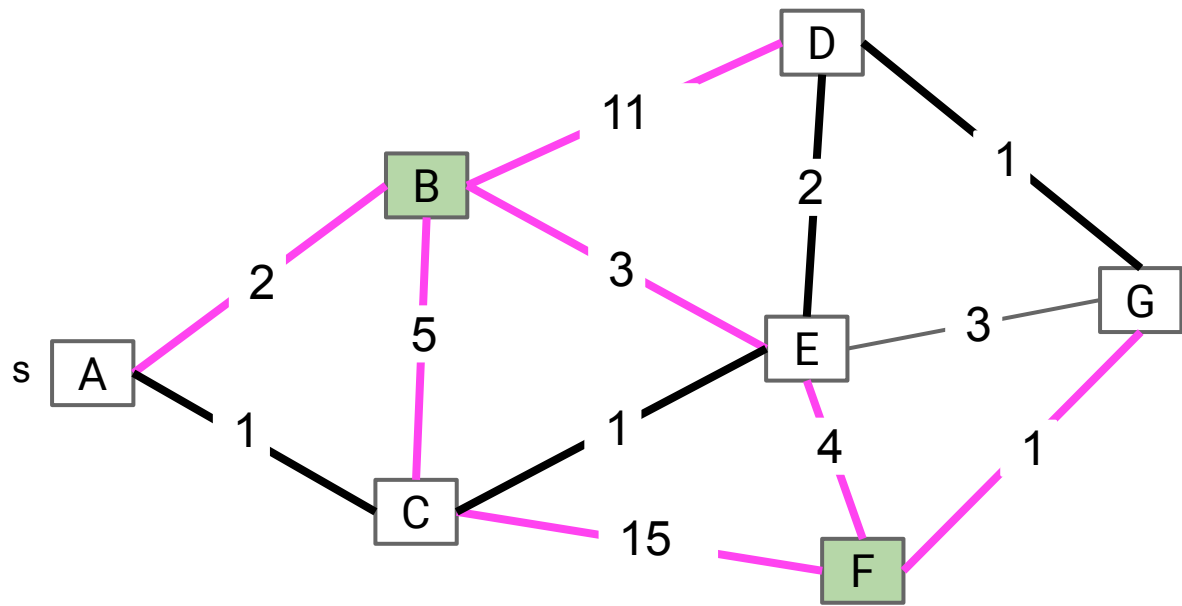


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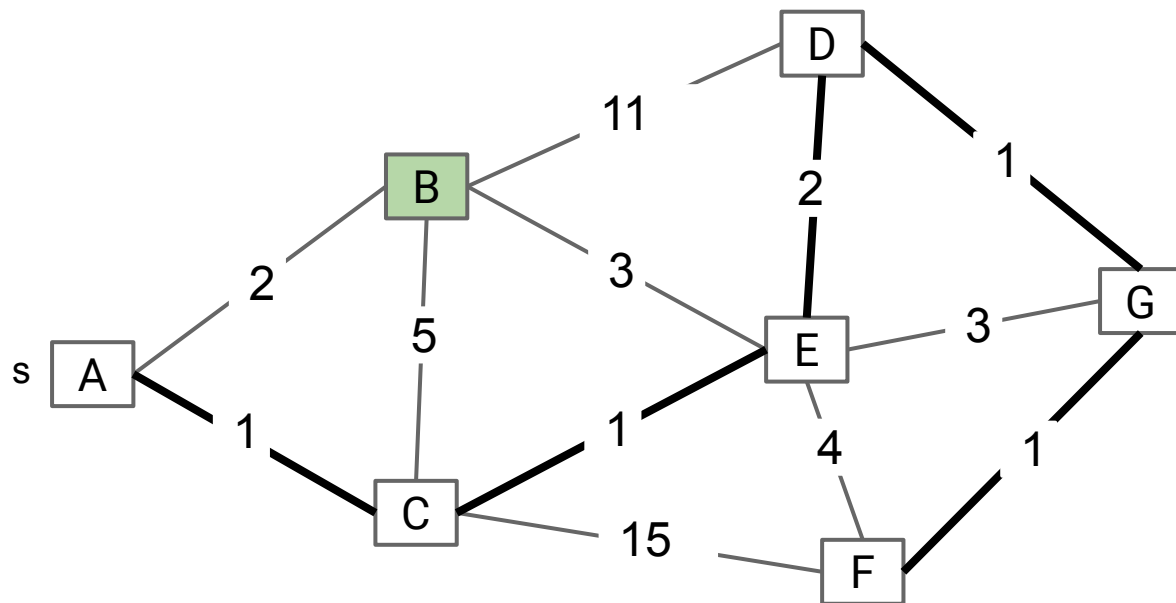


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| B | - |
| C | A |
| D | E |
| E | C |
| F | G |
| G | D |

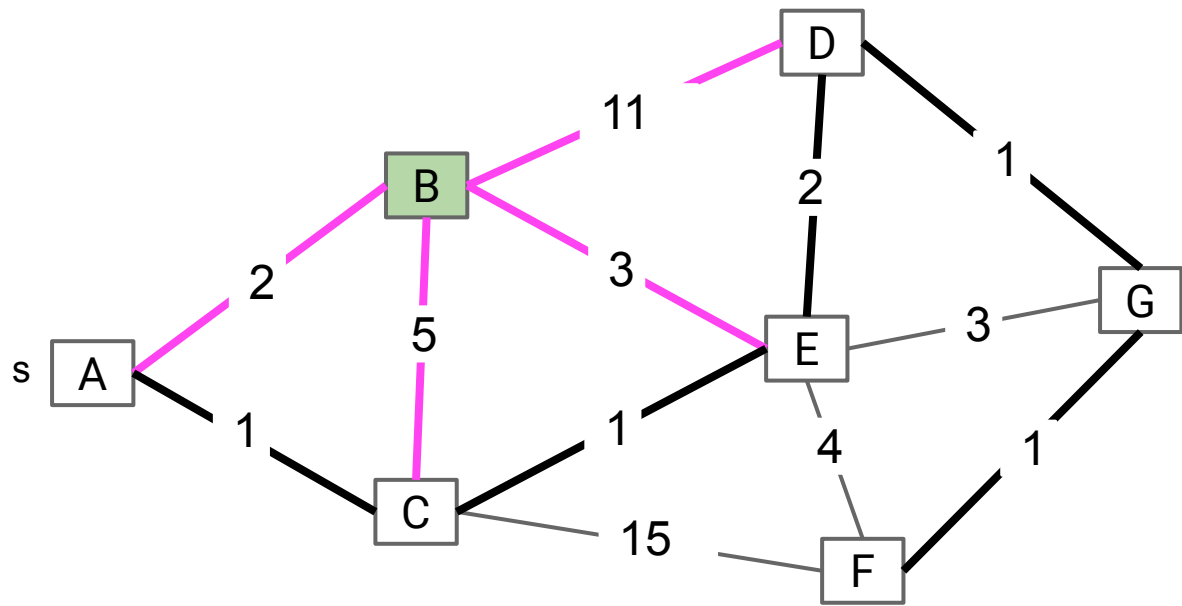


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| E | C |
| F | G |
| G | D |

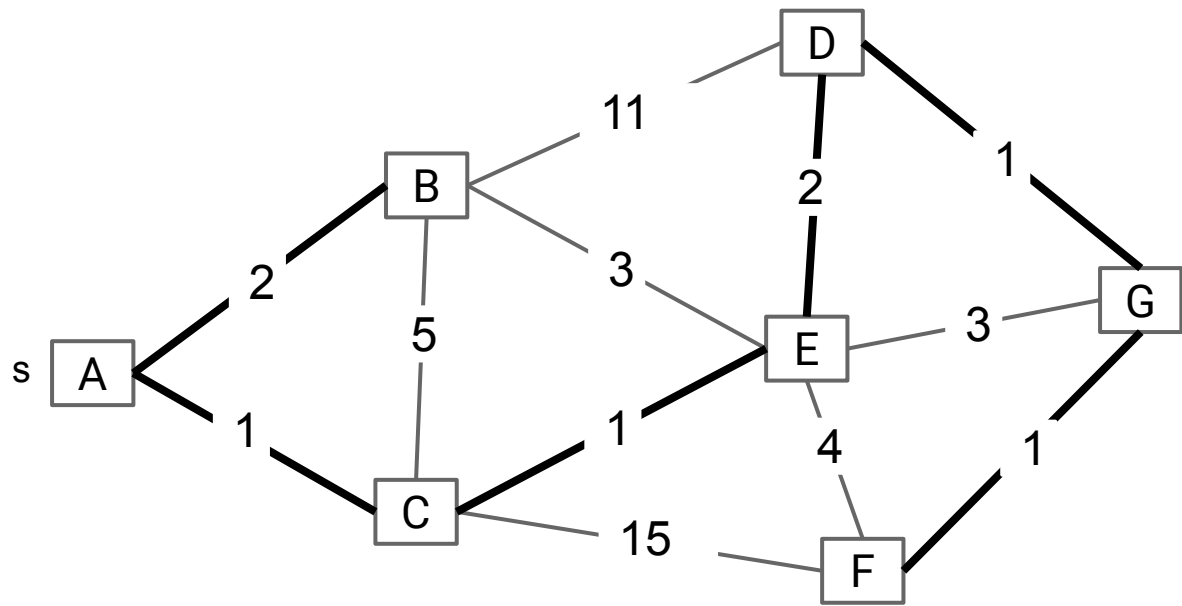


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Start from some arbitrary start node.

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| Node | edgeTo |
|------|--------|
| A | - |
| B | A |
| C | A |
| D | E |
| E | C |
| F | G |
| G | D |



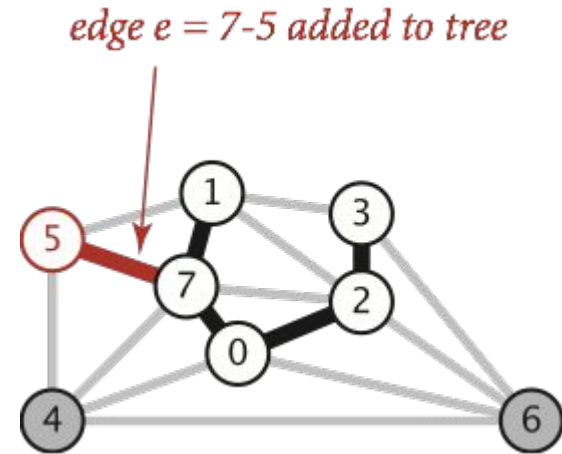
Prim's Algorithm

Start from some arbitrary start node.

- Repeatedly add shortest edge (mark black) that has one node inside the MST under construction.
- Repeat until $V-1$ edges.

Why does Prim's work? Special case of generic algorithm.

- Suppose we add edge $e = v \rightarrow w$.
- Side 1 of cut is all vertices connected to start, side 2 is all the others.
- No crossing edge is black (all connected edges on side 1).
- No crossing edge has lower weight (consider in increasing order).



Optimized Prim's (Demo)

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Prim's Algorithm Implementation

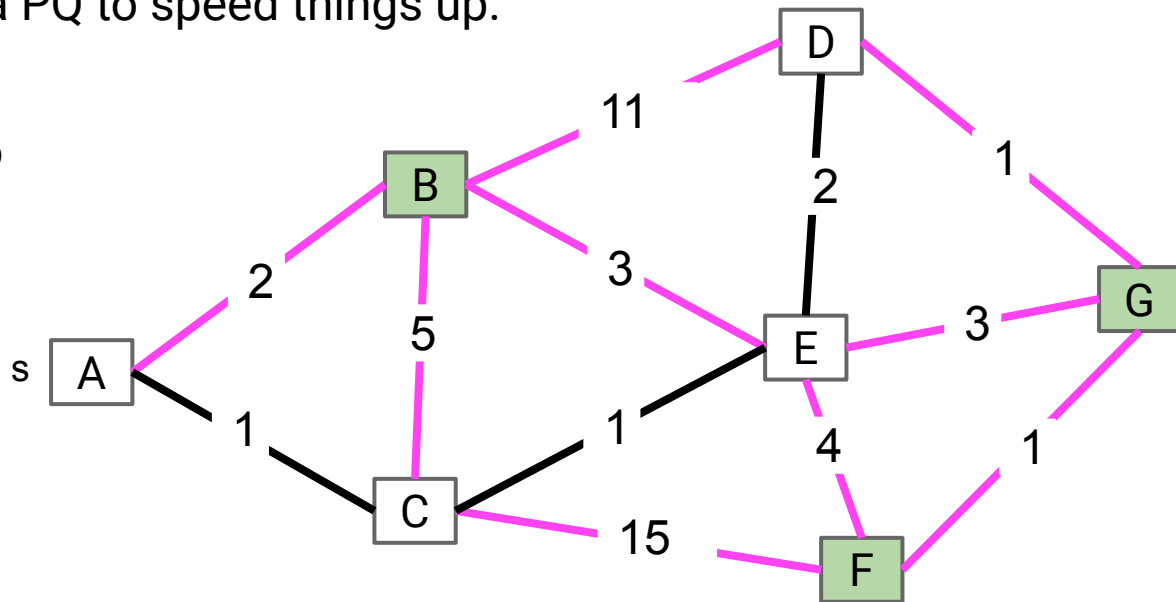
The natural implementation of the conceptual version of Prim's algorithm is highly inefficient.

- Example: Iterating over all purple edges shown is unnecessary and slow.

Can use some cleverness and a PQ to speed things up.

Realistic Implementation Demo

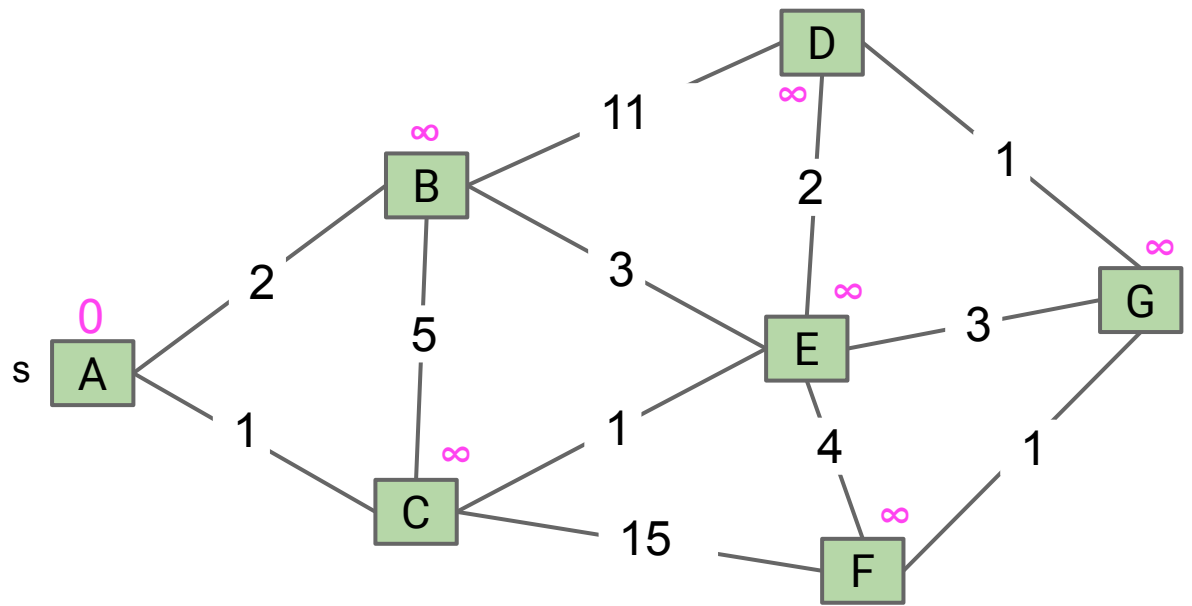
- Very similar to Dijkstra's!



Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|----------|--------|
| A | 0 | - |
| B | ∞ | - |
| C | ∞ | - |
| D | ∞ | - |
| E | ∞ | - |
| F | ∞ | - |
| G | ∞ | - |

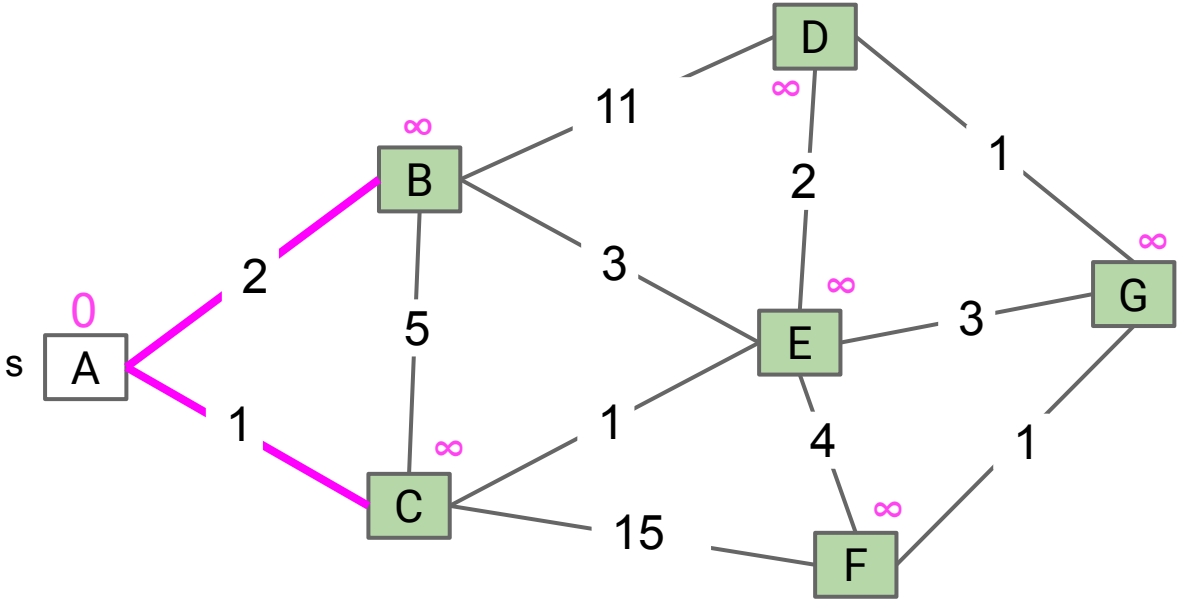


Fringe: [(A: 0), (B: ∞), (C: ∞), (D: ∞), (E: ∞), (F: ∞), (G: ∞)]

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| A | 0 | - |
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| E | ∞ | - |
| F | ∞ | - |
| G | ∞ | - |

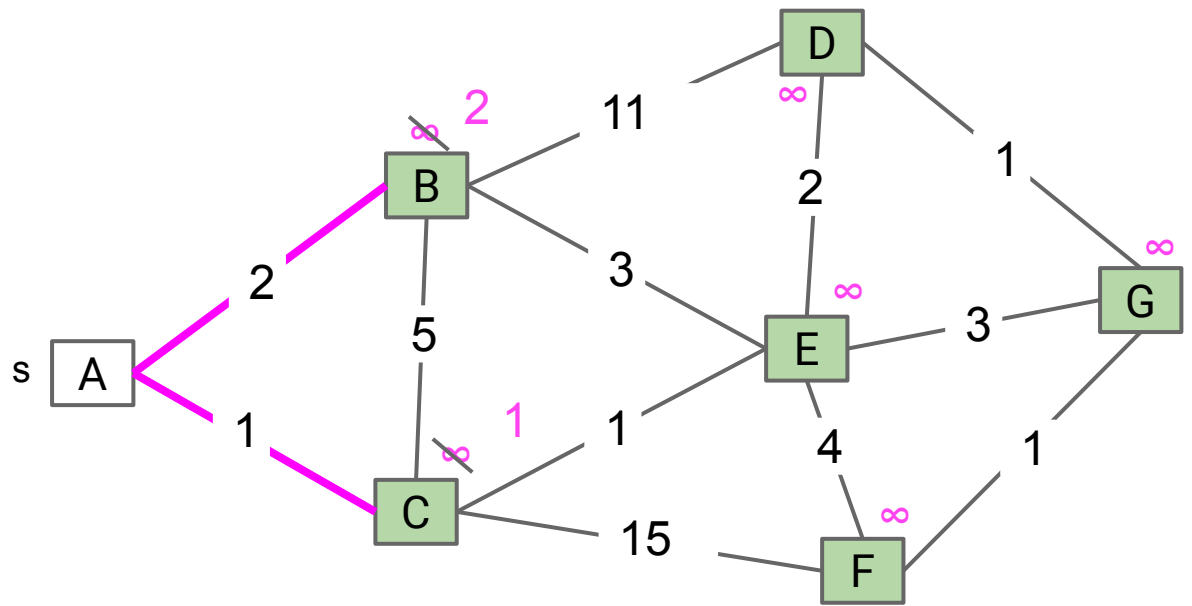


Fringe: [(B: ∞), (C: ∞), (D: ∞), (E: ∞), (F: ∞), (G: ∞)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|----------|--------|
| A | - | - |
| B | 2 | A |
| C | 1 | A |
| D | ∞ | - |
| E | ∞ | - |
| F | ∞ | - |
| G | ∞ | - |

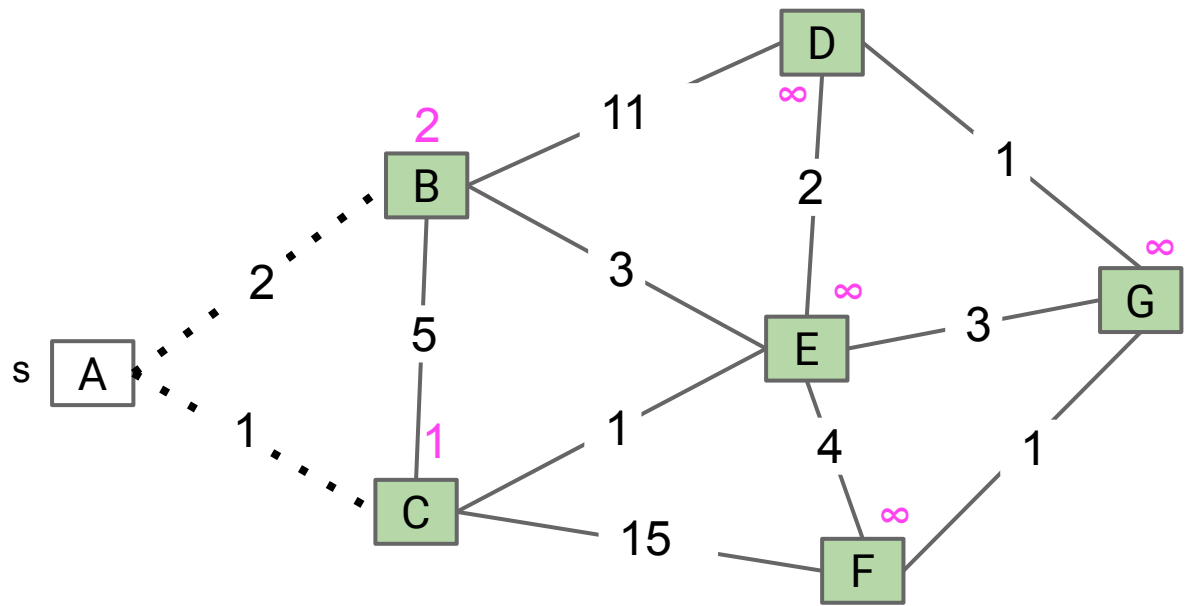


Fringe: [(C: 1), (B: 2), (D: ∞), (E: ∞), (F: ∞), (G: ∞)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | 0 | - |
| B | 2 | A |
| C | 1 | A |
| D | ∞ | - |
| E | ∞ | - |
| F | ∞ | - |
| G | ∞ | - |

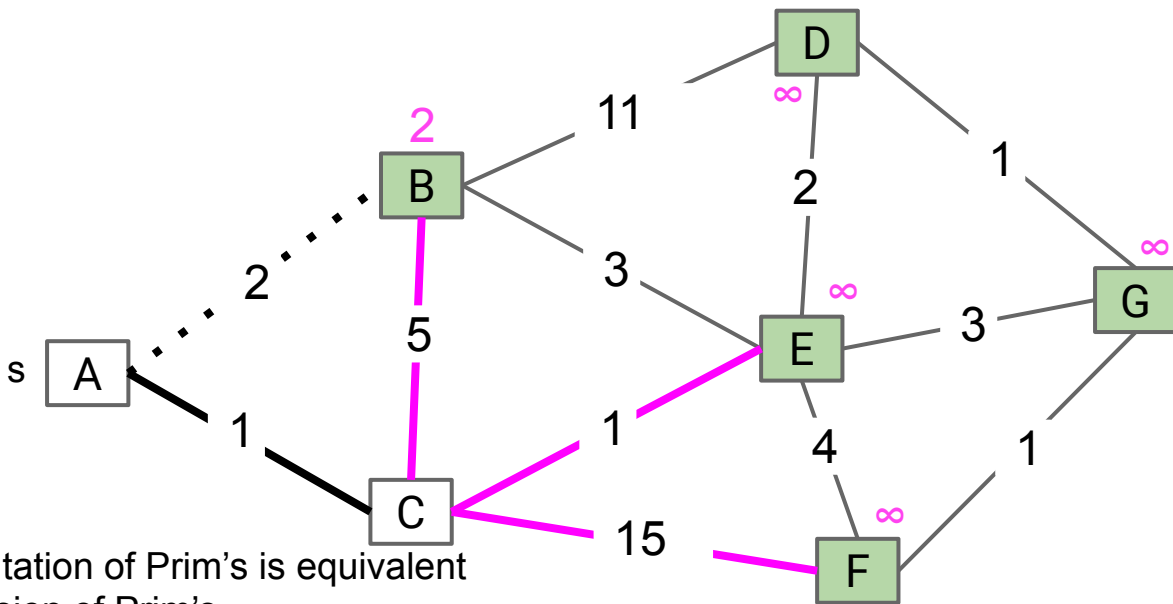


Fringe: [(C: 1), (B: 2), (D: ∞), (E: ∞), (F: ∞), (G: ∞)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|----------|--------|
| A | | - |
| B | 2 | A |
| C | | A |
| D | ∞ | - |
| E | ∞ | - |
| F | ∞ | - |
| G | ∞ | - |



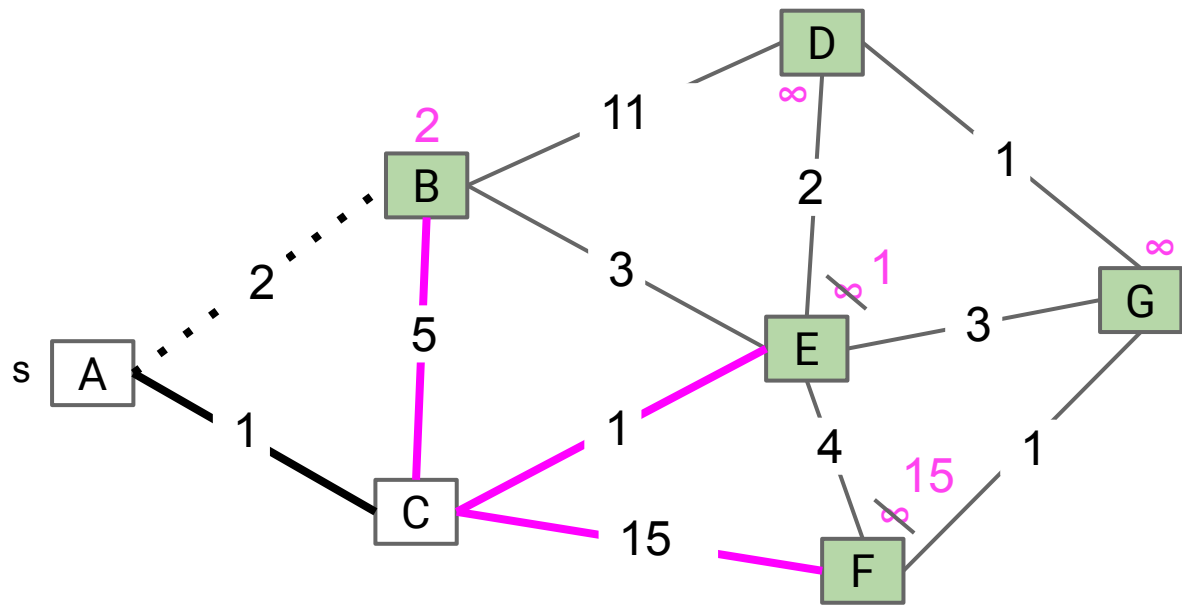
Note: Vertex removal in this implementation of Prim's is equivalent to edge addition in the conceptual version of Prim's.

Fringe: [(B: 2), (D: ∞), (E: ∞), (F: ∞), (G: ∞)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|----------|--------|
| A | | - |
| B | 2 | A |
| C | | A |
| D | ∞ | - |
| E | 1 | C |
| F | 15 | C |
| G | ∞ | - |

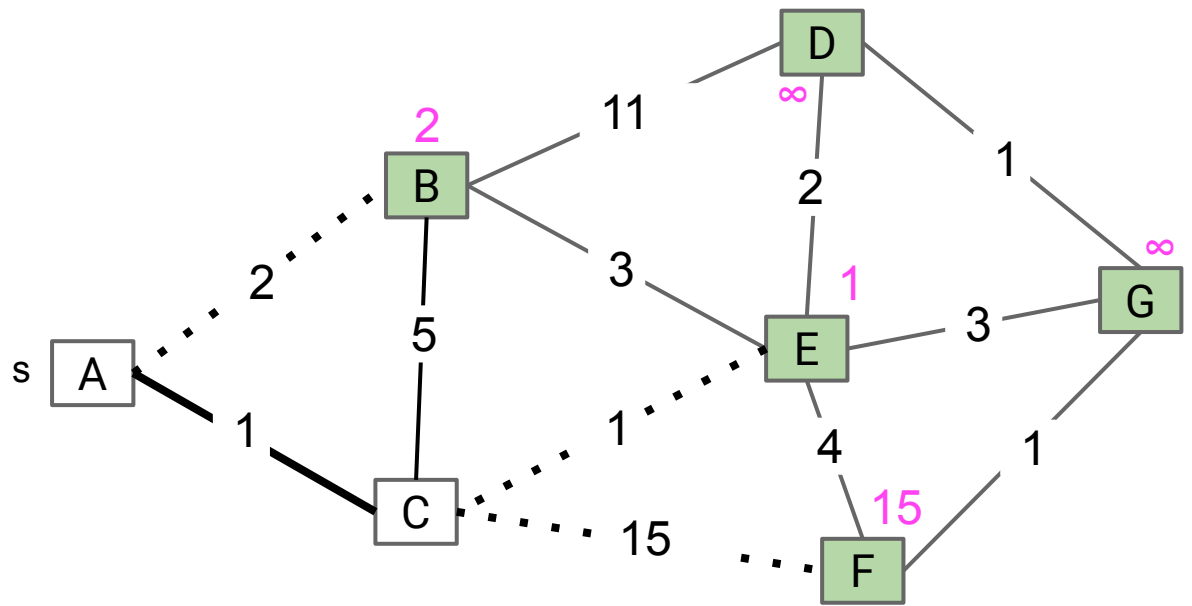


Fringe: [(E: 1), (B: 2), (F: 15), (D: ∞), (G: ∞)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|----------|--------|
| A | | - |
| B | 2 | A |
| C | | A |
| D | ∞ | - |
| E | 1 | C |
| F | 15 | C |
| G | ∞ | - |



Fringe: [(E: 1), (B: 2), (F: 15), (D: ∞), (G: ∞)]

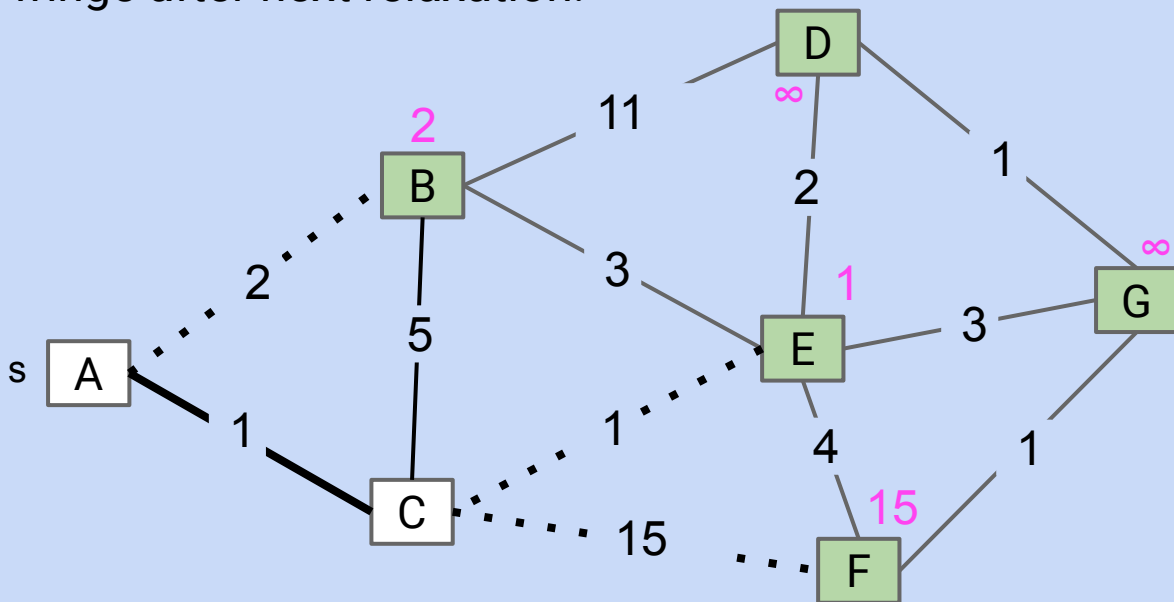
Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.

Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

- Show distTo , edgeTo , and fringe after next relaxation.

| Node | distTo | edgeTo |
|------|----------|--------|
| A | | - |
| B | 2 | A |
| C | | A |
| D | ∞ | - |
| E | 1 | C |
| F | 15 | C |
| G | ∞ | - |

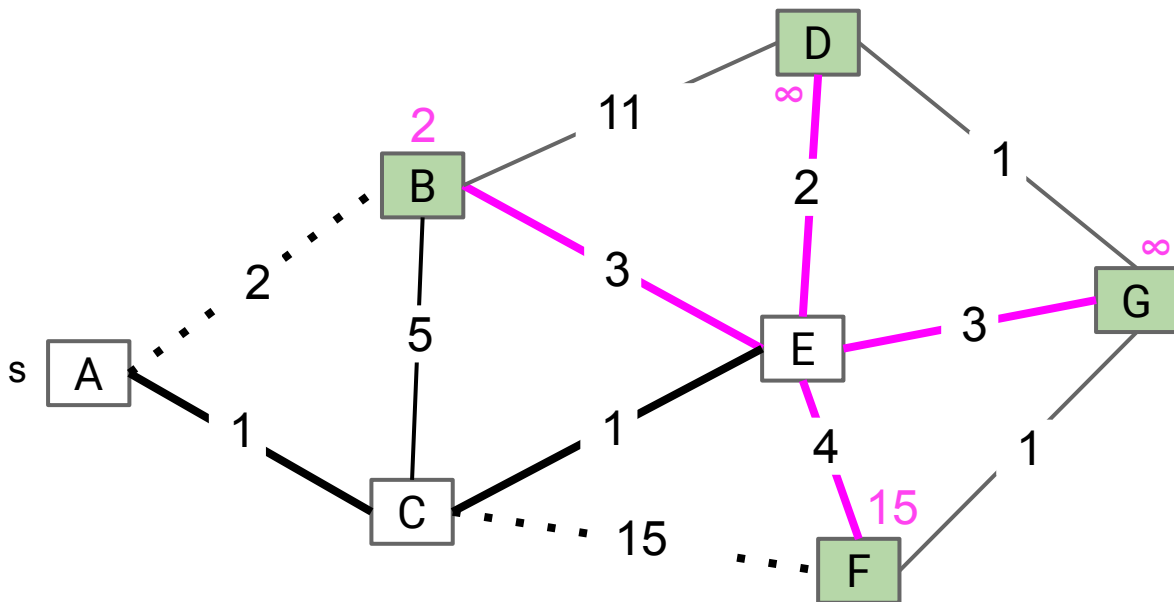


Fringe: [(E: 1), (B: 2), (F: 15), (D: ∞), (G: ∞)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|----------|--------|
| A | | - |
| B | 2 | A |
| C | | A |
| D | ∞ | - |
| E | | C |
| F | 15 | C |
| G | ∞ | - |

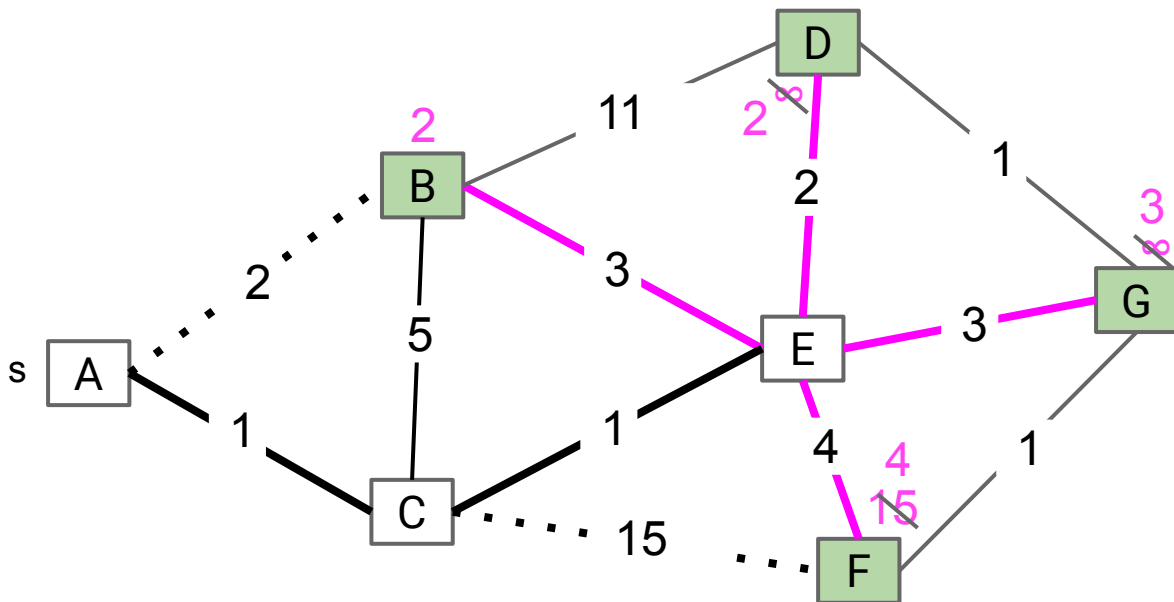


Fringe: [(B: 2), (F: 15), (D: ∞), (G: ∞)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | 2 | A |
| C | | A |
| D | 2 | E |
| E | | C |
| F | 4 | E |
| G | 3 | E |

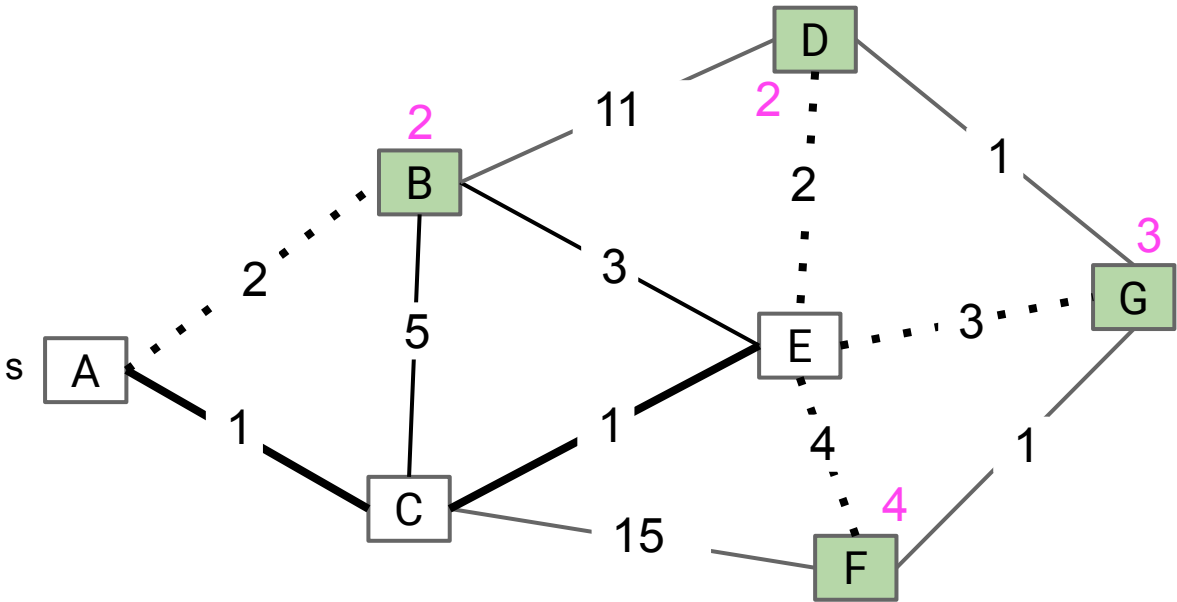


Fringe: [(B: 2), (D: 2), (G: 3), (F: 4)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex *v* from PQ, and relax all edges pointing from *v*.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | 2 | A |
| C | | A |
| D | 2 | E |
| E | | C |
| F | 4 | E |
| G | 3 | E |

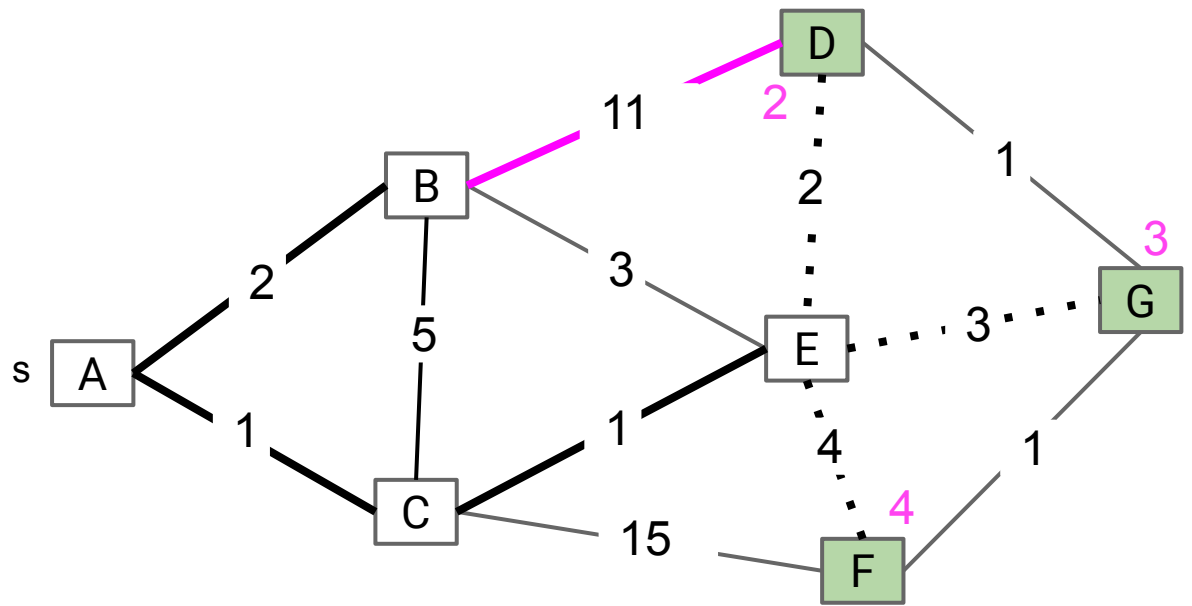


Fringe: [(B: 2), (D: 2), (G: 3), (F: 4)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | 2 | E |
| E | | C |
| F | 4 | E |
| G | 3 | E |



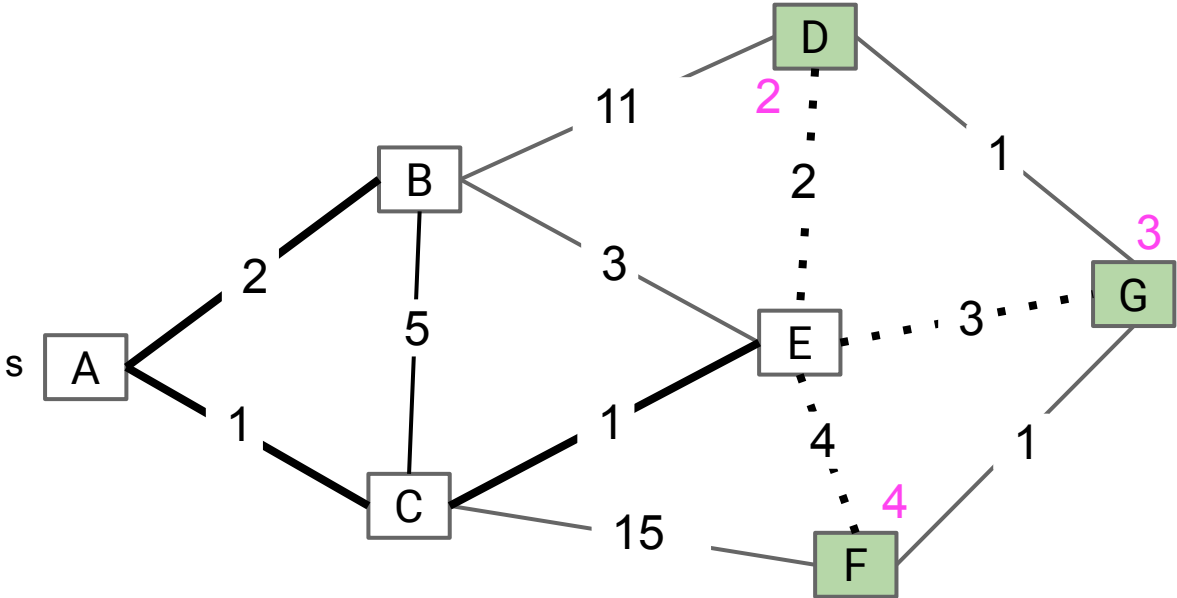
Fringe: [(D: 2), (G: 3), (F: 4)]

No need to consider edges with weight 5 and 3 since other side is already marked!

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | 2 | E |
| E | | C |
| F | 4 | E |
| G | 3 | E |

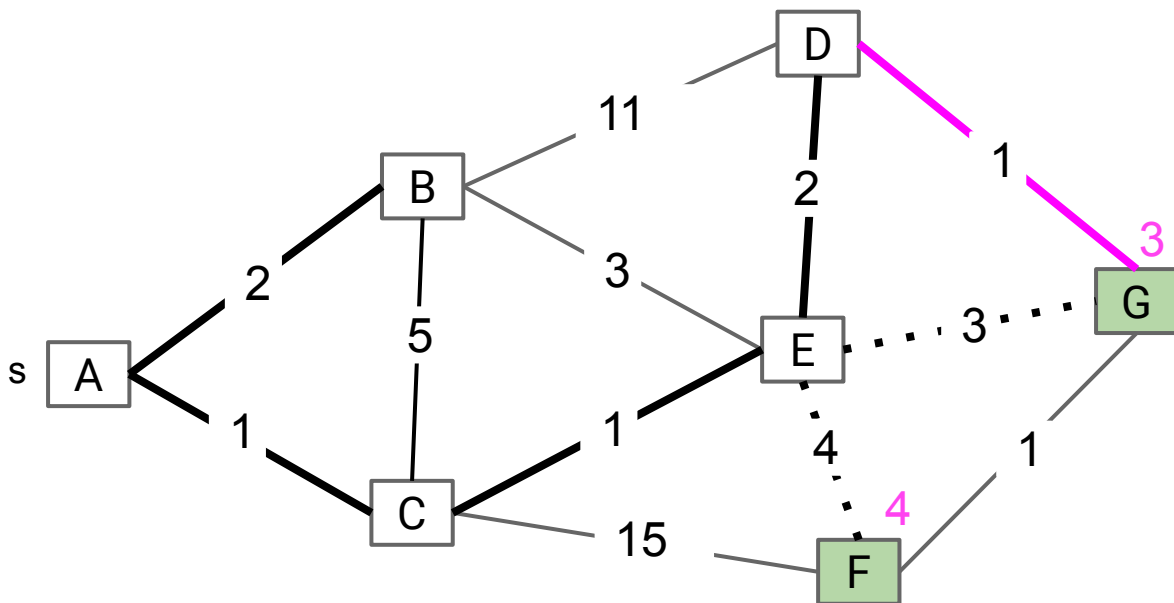


Fringe: [(D: 2), (G: 3), (F: 4)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | | E |
| E | | C |
| F | 4 | E |
| G | 3 | E |

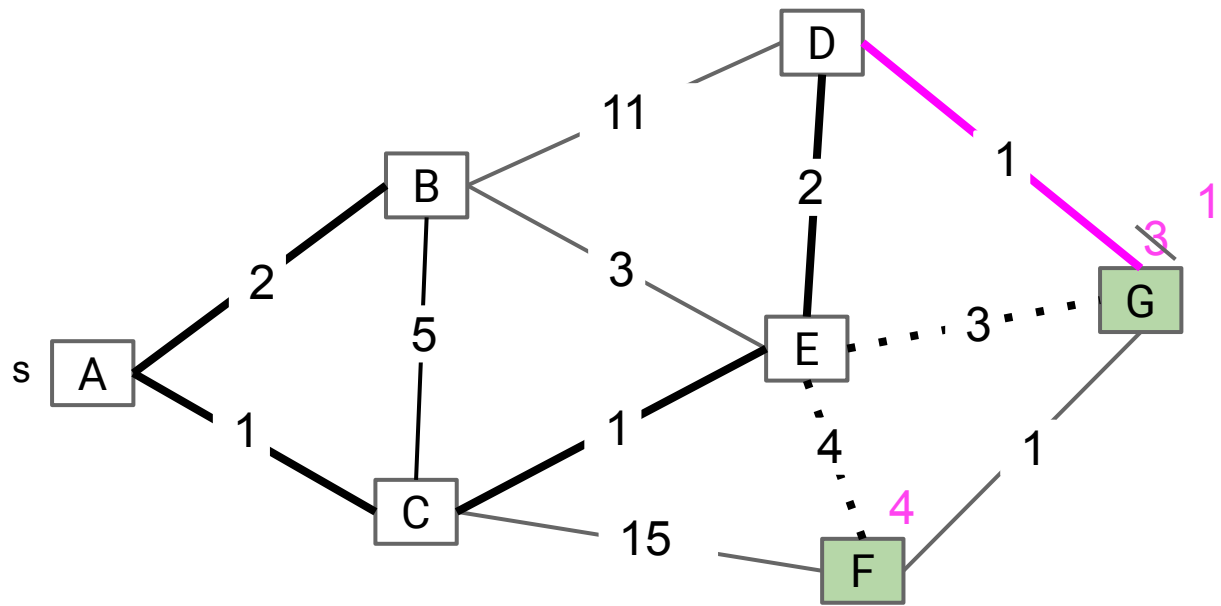


Fringe: [(G: 3), (F: 4)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | | E |
| E | | C |
| F | 4 | E |
| G | 1 | D |



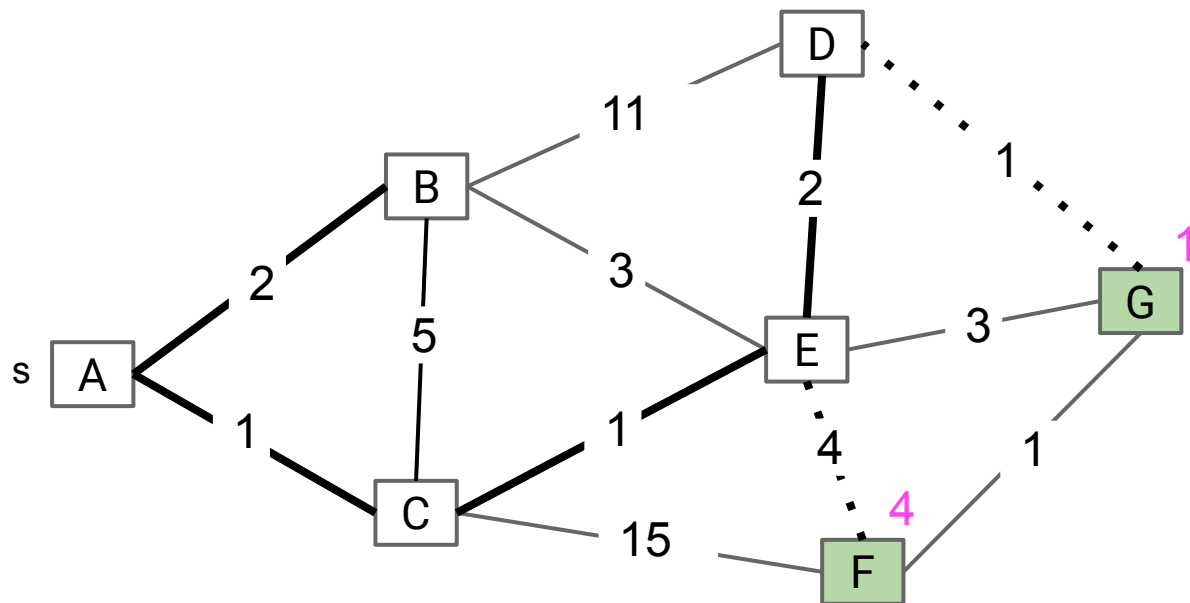
Fringe: [(G: 1), (F: 4)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.

Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v .

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | | E |
| E | | C |
| F | 4 | E |
| G | 1 | D |

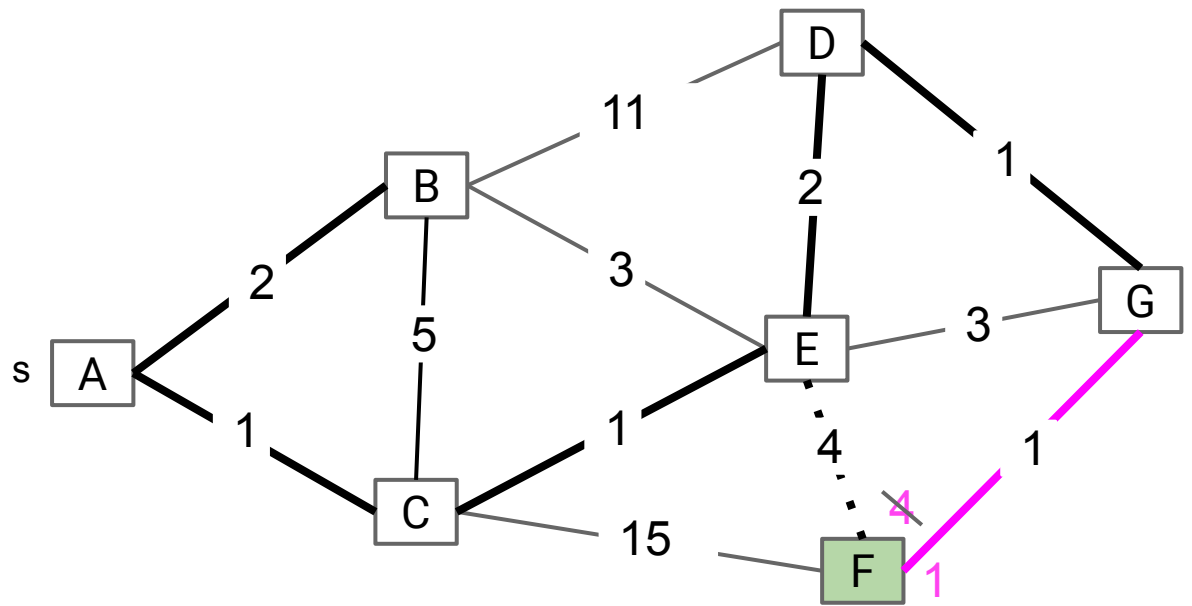


Fringe: [(G: 1), (F: 4)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | | E |
| E | | C |
| F | 1 | G |
| G | | D |

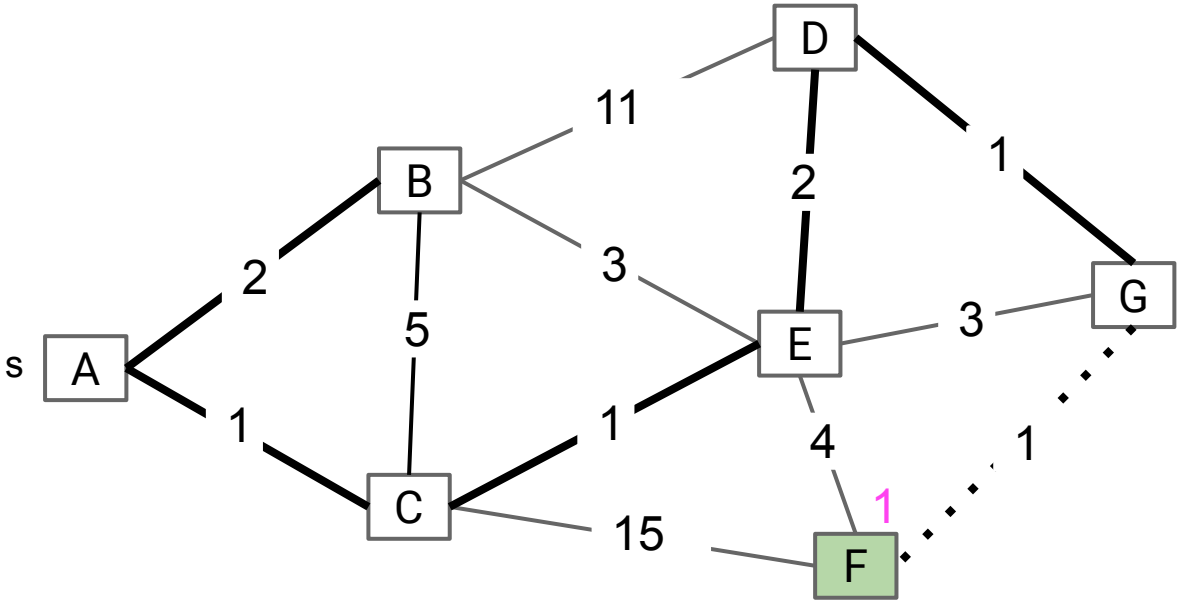


Fringe: [(F: 1)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | | E |
| E | | C |
| F | 1 | G |
| G | | D |

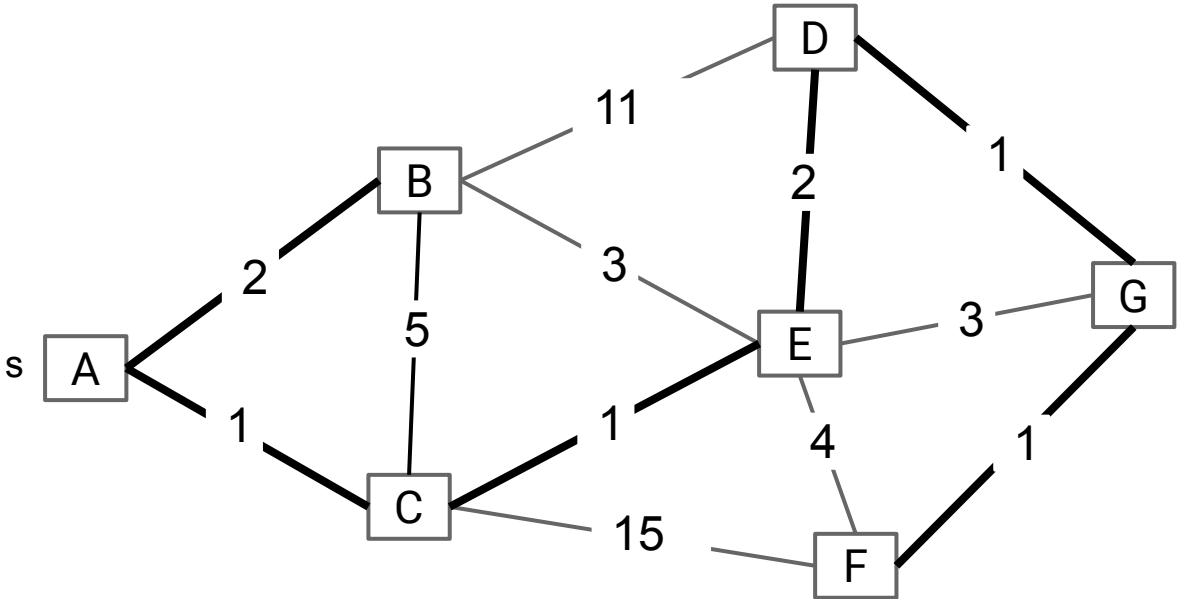


Fringe: [(F: 1)]

Prim's Demo

Insert all vertices into fringe PQ, storing vertices in order of distance from tree.
Repeat: Remove (closest) vertex v from PQ, and relax all edges pointing from v.

| Node | distTo | edgeTo |
|------|--------|--------|
| A | | - |
| B | | A |
| C | | A |
| D | | E |
| E | | C |
| F | | G |
| G | | D |



Fringe: []

Prim's Algorithm Code and Runtime

Lecture 25, CS61B, Spring 2024

Graph Problem Warmup
Minimum Spanning Trees

- Intro
- The Cut Property

Prim's Algorithm

- Basic Prim's (Demo)
- Optimized Prim's (Demo)
- **Prim's Algorithm Code and Runtime**

Kruskal's Algorithm:

- Basic Kruskal's (Demo)
- Optimized Kruskal's (Demo)
- Kruskal's vs. Prim's
- Kruskal's Algorithm Code and Runtime

Prim's and Dijkstra's algorithms are exactly the same, except Dijkstra's considers "distance from the source", and Prim's considers "distance from the tree."

Visit order:

- Dijkstra's algorithm visits vertices in order of distance from the source.
- Prim's algorithm visits vertices in order of distance from the MST under construction.

Relaxation:

- Relaxation in Dijkstra's considers an edge better based on distance to source.
- Relaxation in Prim's considers an edge better based on distance to tree.

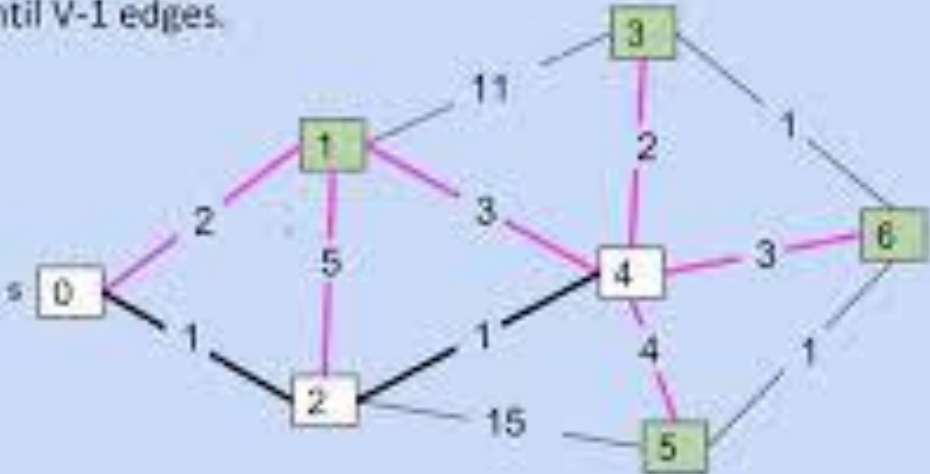
Prim's Demo (Conceptual)



Start from some arbitrary start node.

- Add shortest edge (mark black) that has one node inside the MST under construction. Repeat until $V-1$ edges.

| # | edgeTo |
|---|--------|
| 0 | - |
| 1 | - |
| 2 | 0 |
| 3 | - |
| 4 | 2 |
| 5 | - |
| 6 | - |



Which edge is added next?

Prim's Implementation (Pseudocode, 1/2)

```
public class PrimMST {  
    public PrimMST(EdgeWeightedGraph G) {  
        edgeTo = new Edge[G.V()];  
        distTo = new double[G.V()];  
        marked = new boolean[G.V()];  
        fringe = new SpecialPQ<Double>(G.V());  
  
        distTo[s] = 0.0;  
        setDistancesToInfinityExceptS(s);  
        insertAllVertices(fringe);  
  
        /* Get vertices in order of distance from tree. */  
        while (!fringe.isEmpty()) {  
            int v = fringe.delMin();  
            scan(G, v);  
        }  
    }  
    ...  
}
```

Fringe is ordered by distTo tree. Must be a specialPQ like Dijkstra's.

Get vertex closest to tree that is unvisited.

Scan means to consider all of a vertices outgoing edges.

Prim's Implementation (Pseudocode, 2/2)

```
while (!fringe.isEmpty()) {  
    int v = fringe.delMin();  
    scan(G, v);  
}
```

Important invariant, fringe must be ordered by current best known distance from tree.

```
private void scan(EdgeWeightedGraph G, int v) {  
    marked[v] = true;  
    for (Edge e : G.adj(v)) {  
        int w = e.other(v);  
        if (marked[w]) { continue; }  
        if (e.weight() < distTo[w]) {  
            distTo[w] = e.weight();  
            edgeTo[w] = e;  
            pq.decreasePriority(w, distTo[w]);  
        }  
    }  
}
```

Vertex is closest, so add to MST.

Already in MST, so go to next edge.

Better path to a particular vertex found, so update current best known for that vertex.

```
while (!fringe.isEmpty()) {  
    int v = fringe.delMin();  
    scan(G, v);  
}
```

```
private void scan(EdgeWeightedGraph G, int v) {  
    marked[v] = true;  
    for (Edge e : G.adj(v)) {  
        int w = e.other(v);  
        if (marked[w]) { continue; }  
        if (e.weight() < distTo[w]) {  
            distTo[w] = e.weight();  
            edgeTo[w] = e;  
            pq.decreasePriority(w, distTo[w]);  
        }  
    }  
}
```

What is the runtime of Prim's algorithm?

- Assume all PQ operations take $O(\log(V))$ time.
- Give your answer in Big O notation.

Priority Queue operation count, assuming binary heap based PQ:

- Insertion: V , each costing $O(\log V)$ time.
- Delete-min: V , each costing $O(\log V)$ time.
- Decrease priority: $O(E)$, each costing $O(\log V)$ time.
 - Operation not discussed in lecture, but it was in lab 10.

Overall runtime: $O(V \cdot \log(V) + V \cdot \log(V) + E \cdot \log(V))$.

- Assuming $E > V$, this is just $O(E \log V)$ (Same as Dijkstra's).

| | # Operations | Cost per operation | Total cost |
|---------------------|--------------|--------------------|---------------|
| PQ add | V | $O(\log V)$ | $O(V \log V)$ |
| PQ delMin | V | $O(\log V)$ | $O(V \log V)$ |
| PQ decreasePriority | $O(E)$ | $O(\log V)$ | $O(E \log V)$ |

Basic Kruskal's (Demo)

Lecture 25, CS61B, Spring 2024

Graph Problem Warmup
Minimum Spanning Trees

- Intro
- The Cut Property

Prim's Algorithm

- Basic Prim's (Demo)
- Optimized Prim's (Demo)
- Prim's Algorithm Code and Runtime

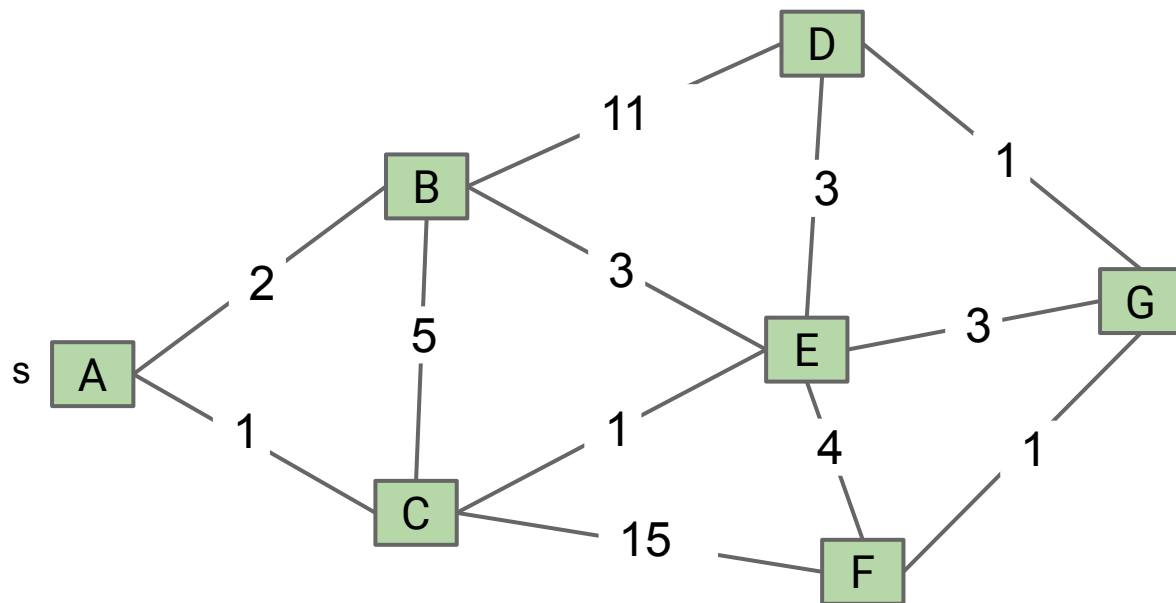
Kruskal's Algorithm:

- **Basic Kruskal's (Demo)**
- Optimized Kruskal's (Demo)
- Kruskal's vs. Prim's
- Kruskal's Algorithm Code and Runtime

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until $V-1$ edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |



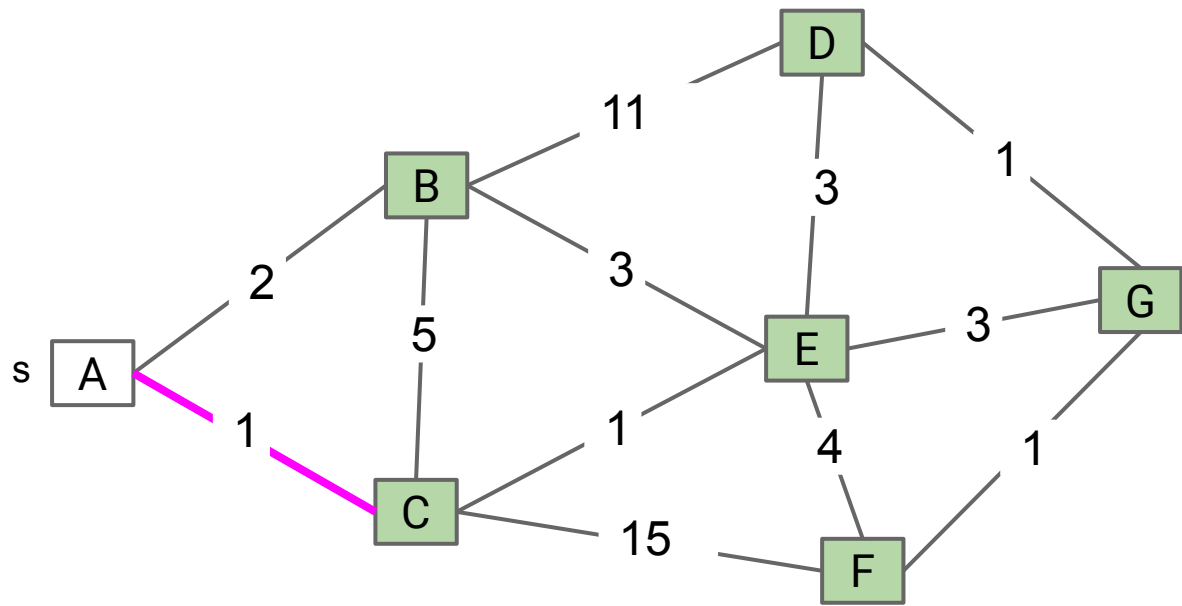
MST: []

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle?



MST: []

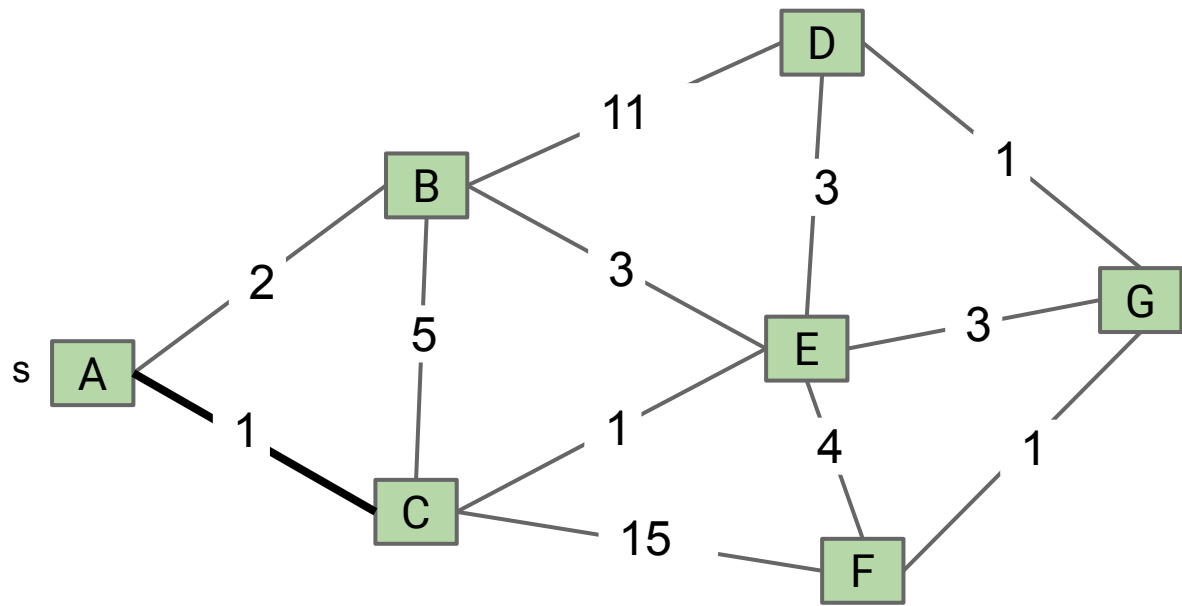
White and green colorings for vertices show cut being implicitly utilized by Kruskal's algorithm.

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|----------------|--------------|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? No.



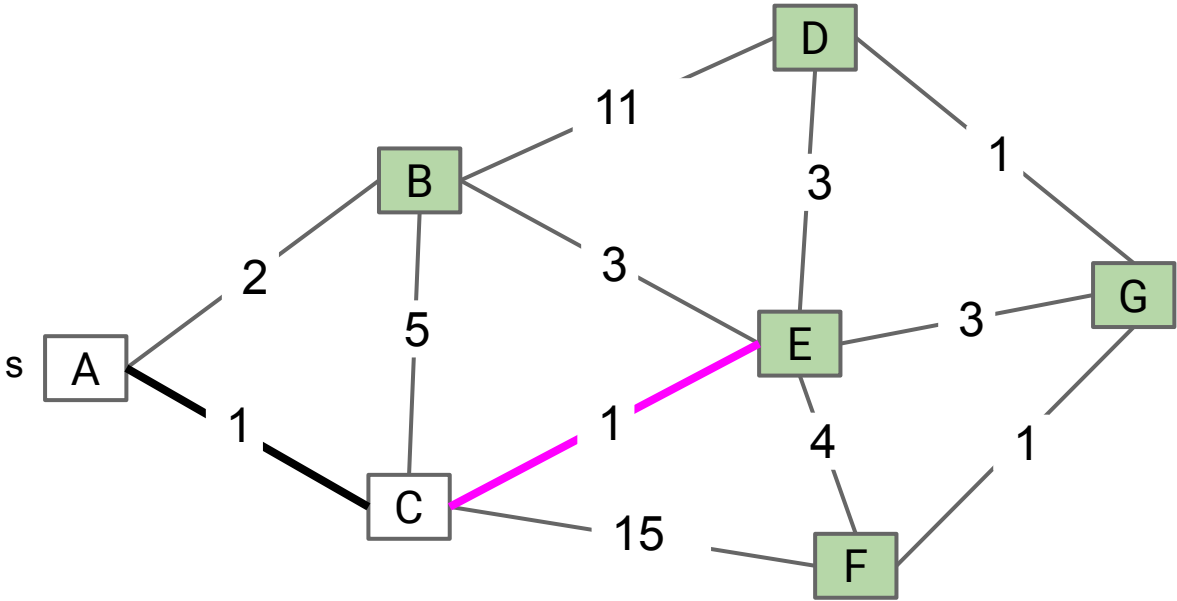
MST: [A-C]

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? No.



MST: [A-C]

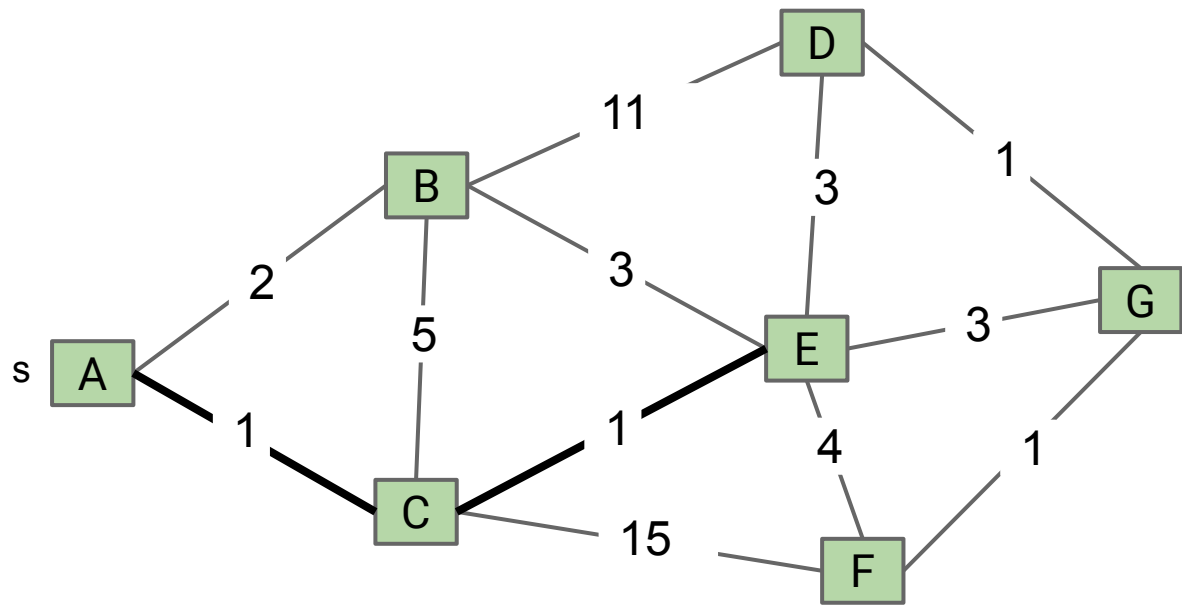
White and green colorings for vertices show cut being implicitly utilized by Kruskal's algorithm.

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|----------------|--------------|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? No.



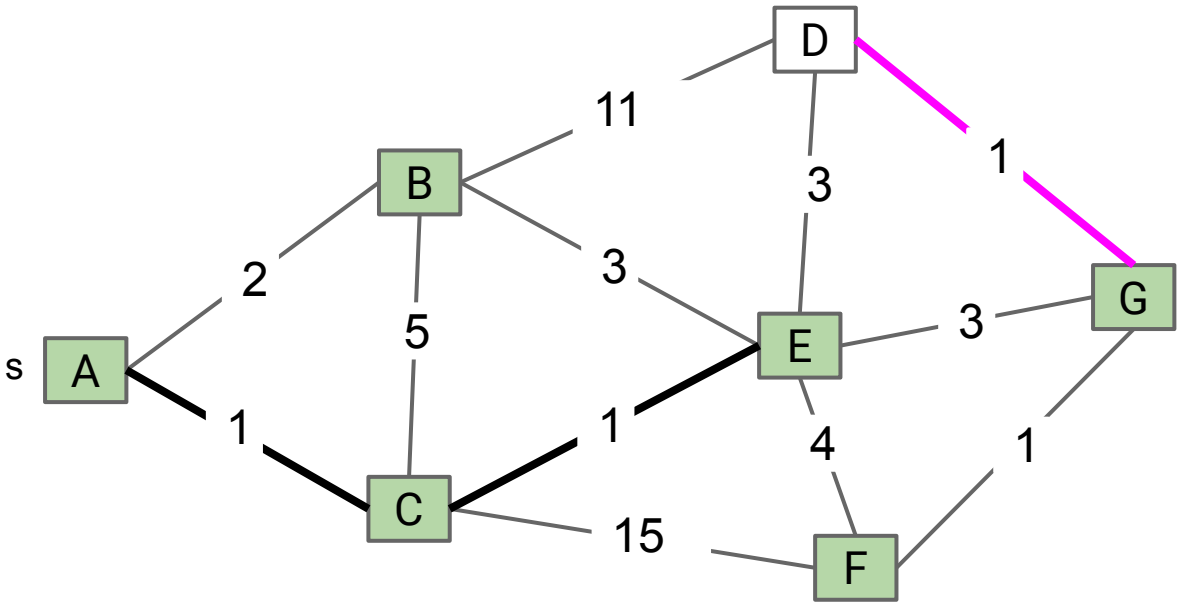
MST: [A-C, C-E]

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle?



MST: [A-C, C-E]

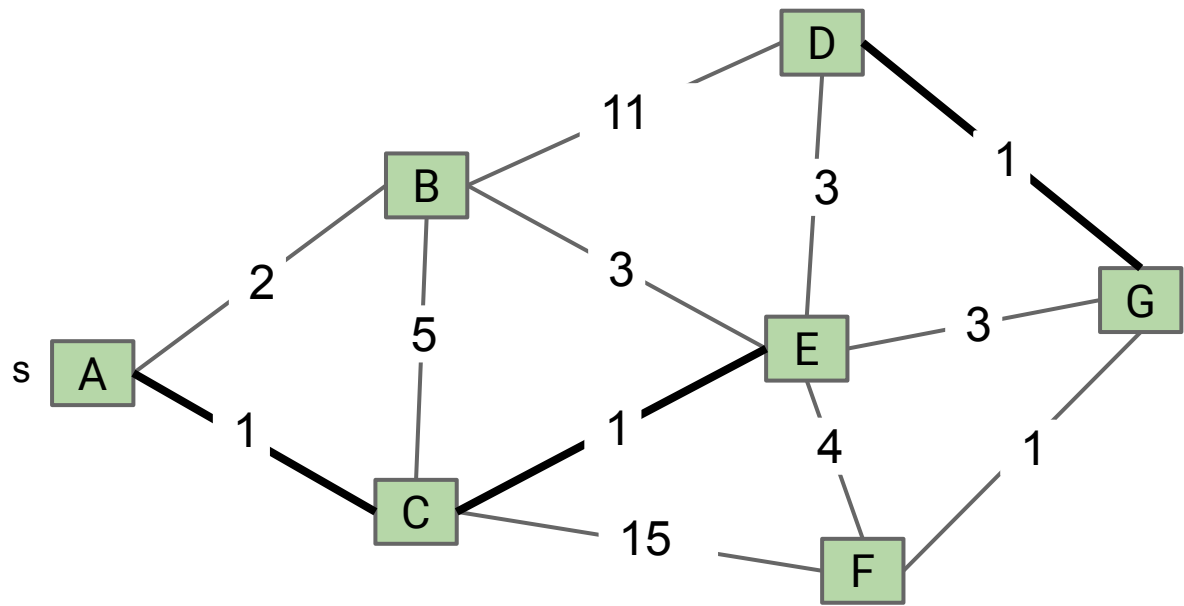
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Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|----------------|--------------|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? No.



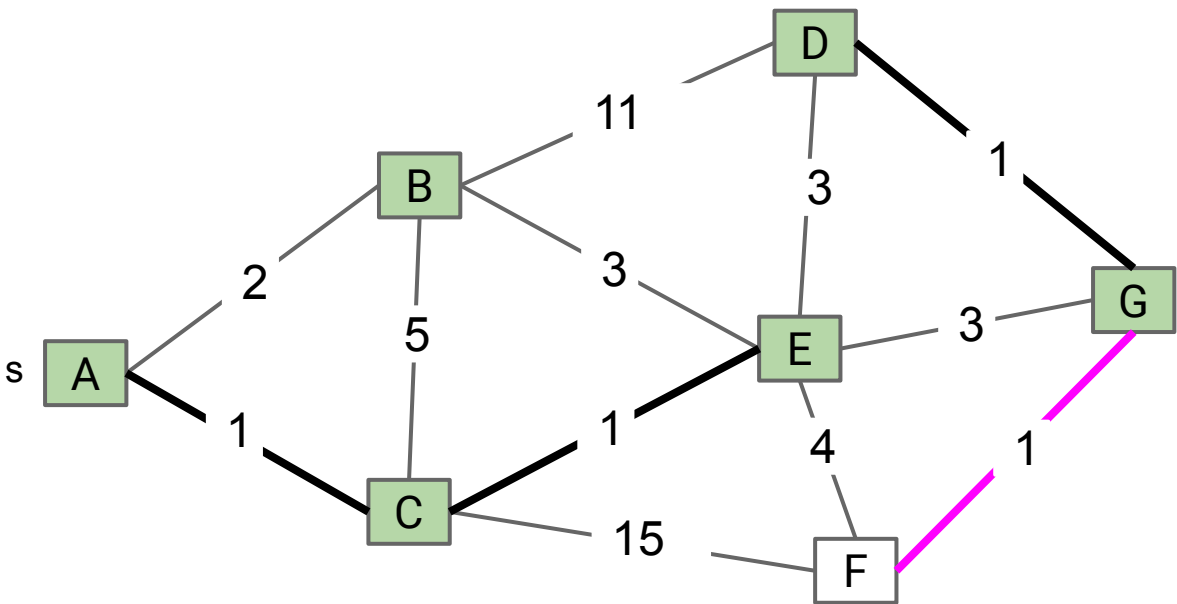
MST: [A-C, C-E, D-G]

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle?



MST: [A-C, C-E, D-G]

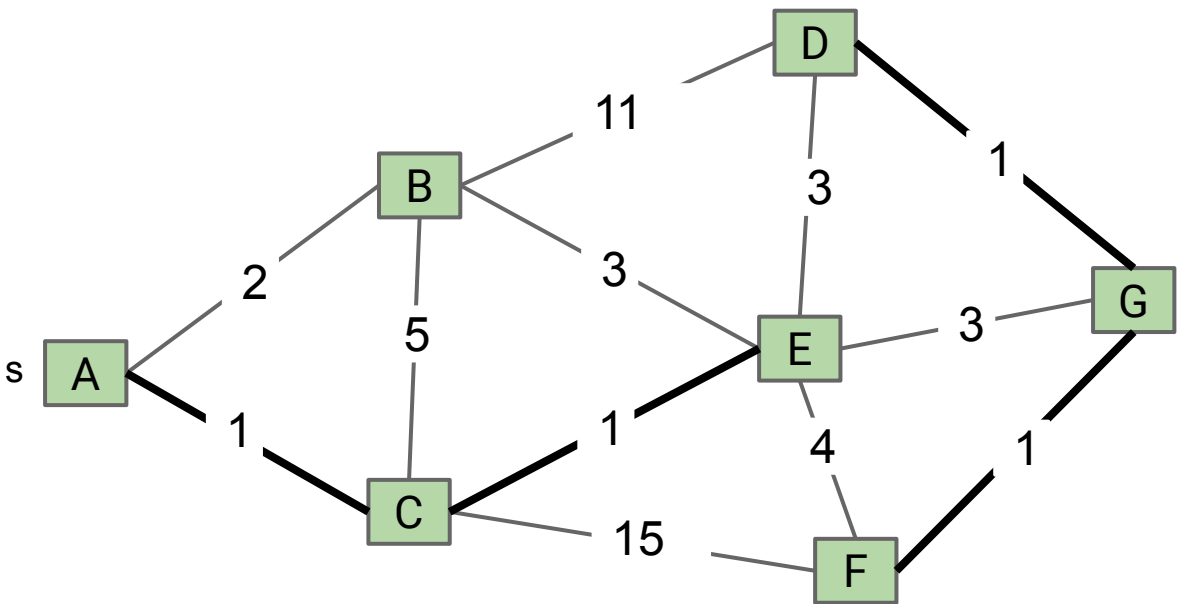
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Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|----------------|--------------|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? No.



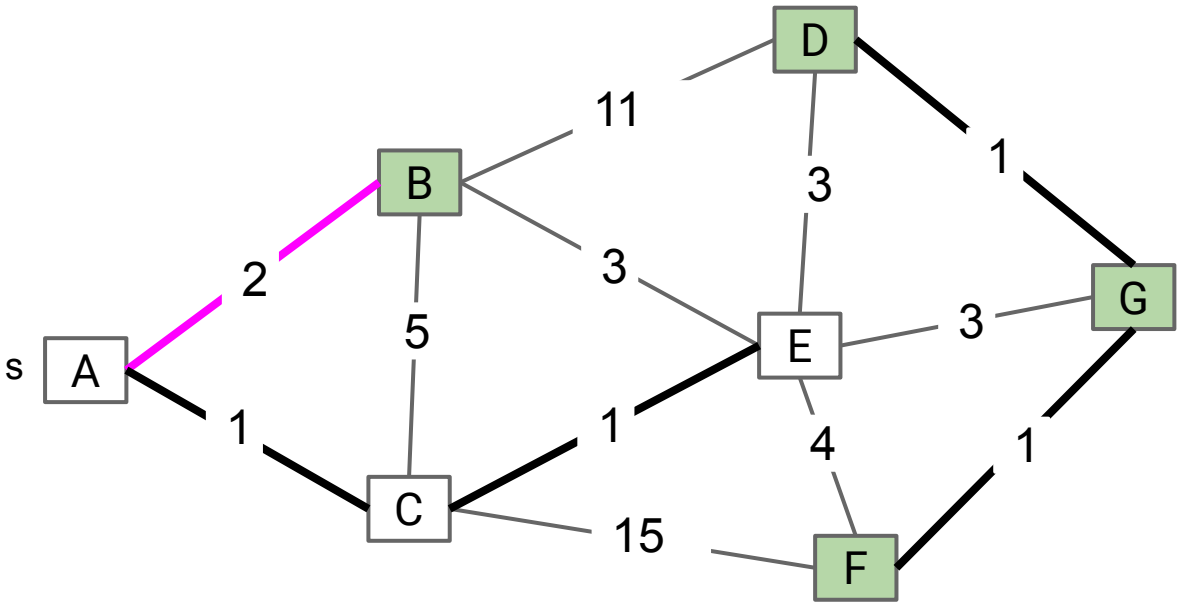
MST: [A-C, C-E, D-G, F-G]

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle?



MST: [A-C, C-E, D-G, F-G]

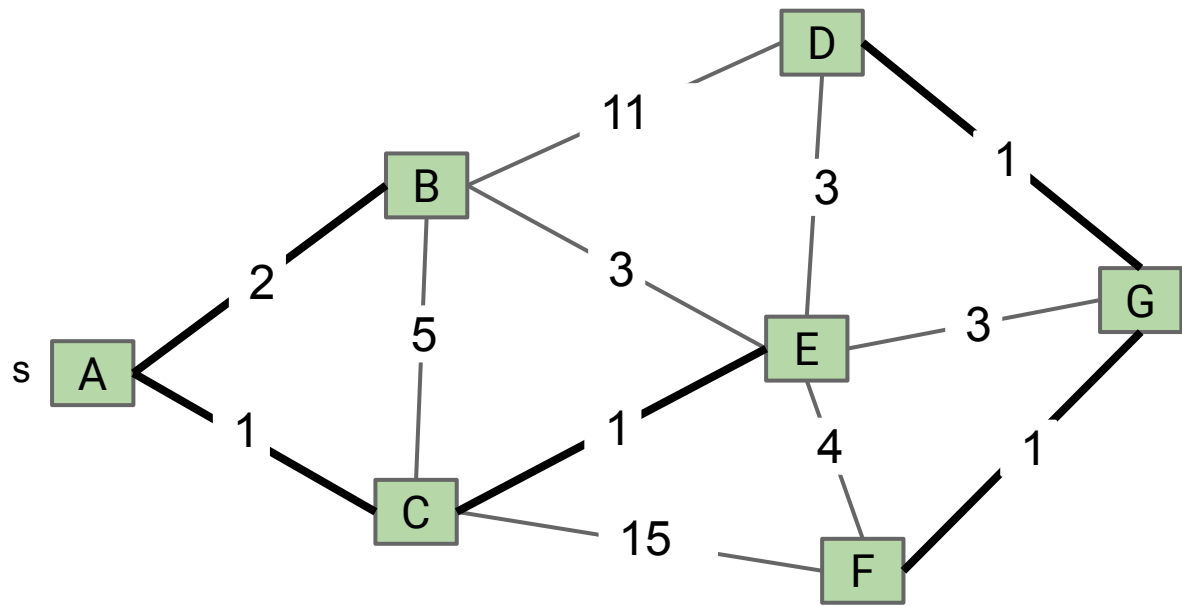
White and green colorings for vertices show cut being implicitly utilized by Kruskal's algorithm.

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|----------------|--------------|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? No.



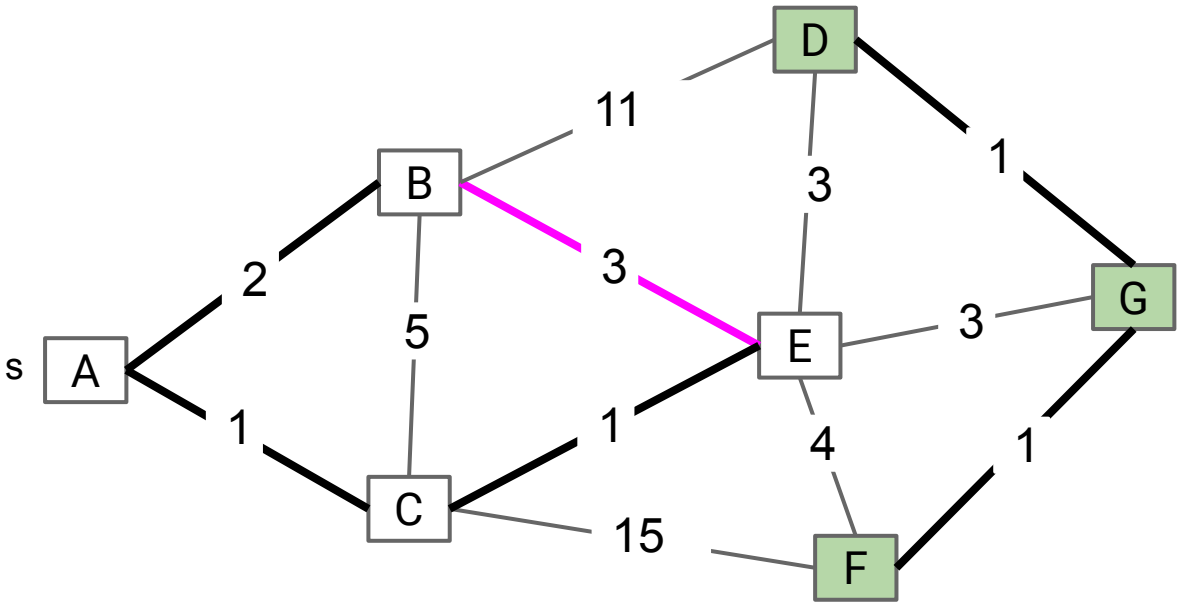
MST: [A-C, C-E, D-G, F-G, A-B]

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle?



White and green colorings for vertices show cut being implicitly utilized by Kruskal's algorithm.

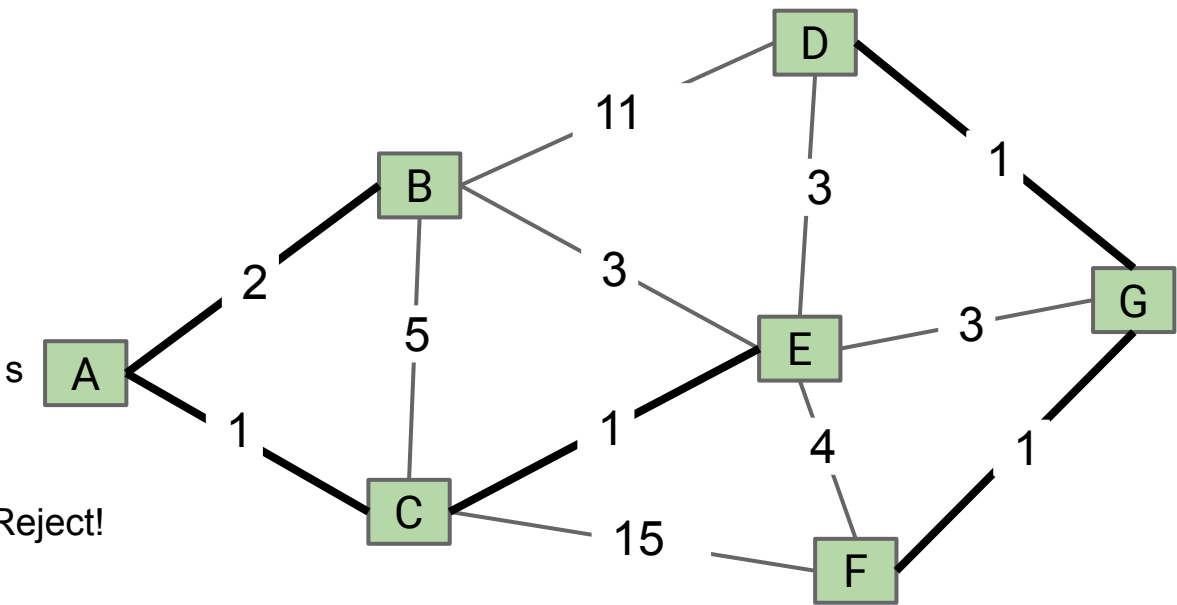
MST: [A-C, C-E, D-G, F-G, A-B]

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|----------------|--------------|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? Yes. Reject!



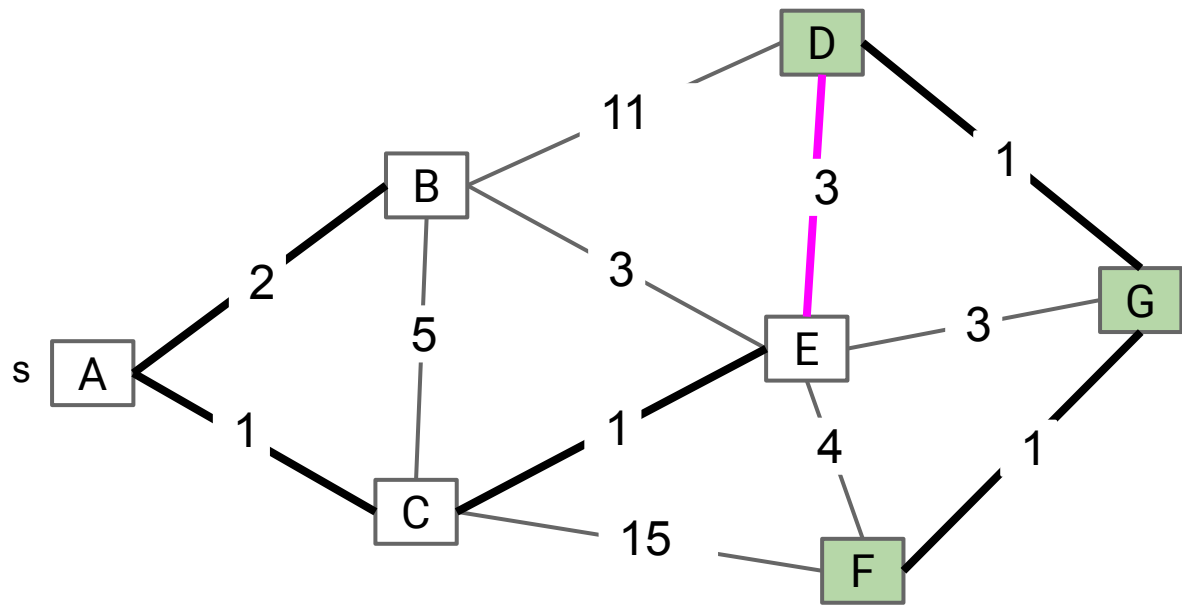
MST: [A-C, C-E, D-G, F-G, A-B]

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle?



MST: [A-C, C-E, D-G, F-G, A-B]

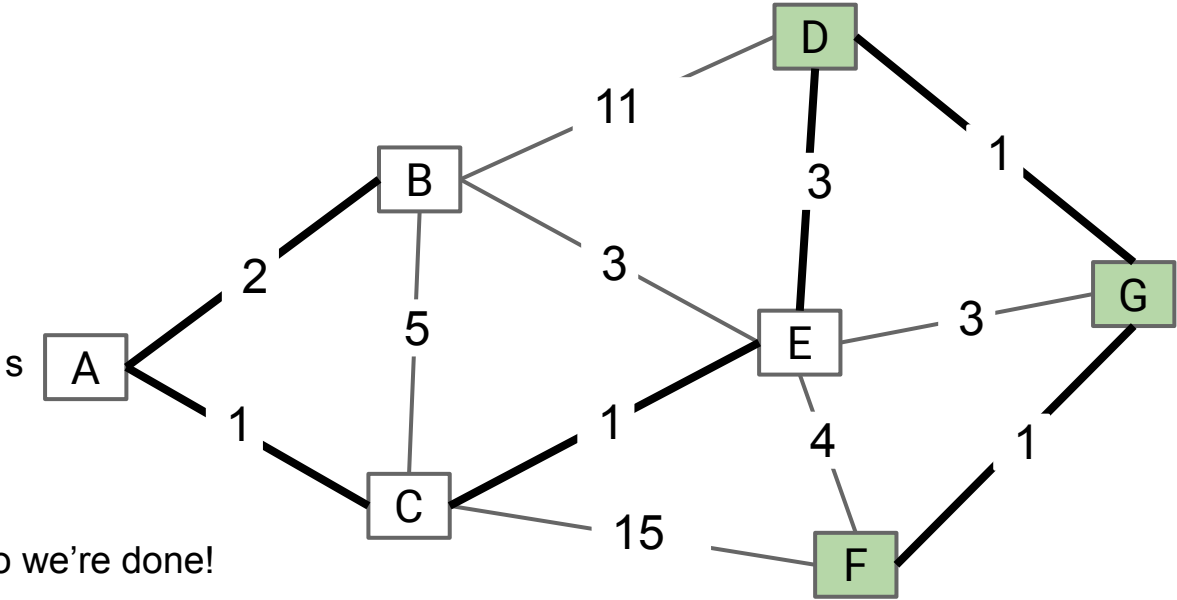
White and green colorings for vertices show cut being implicitly utilized by Kruskal's algorithm.

Kruskal's Demo

Consider edges in order of increasing weight. Add to MST unless a cycle is created. Repeat until V-1 edges.

| | |
|-----|----|
| A-C | 1 |
| C-E | 1 |
| D-G | 1 |
| F-G | 1 |
| A-B | 2 |
| E-B | 3 |
| D-E | 3 |
| G-E | 3 |
| E-F | 4 |
| B-C | 5 |
| B-D | 11 |
| C-F | 15 |

Cycle? No.
V-1 edges, so we're done!

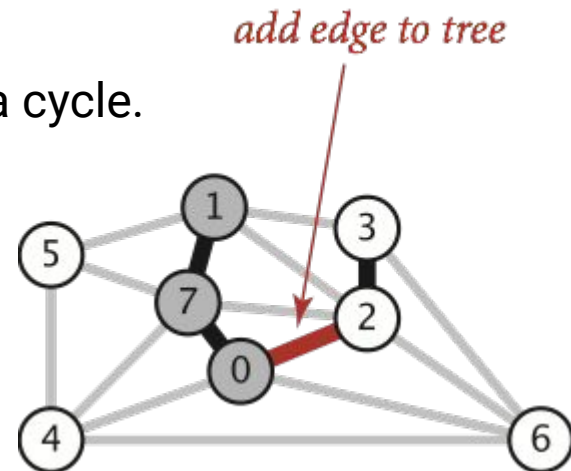


MST: [A-C, C-E, D-G, F-G, A-B, D-E]

Kruskal's Algorithm

Initially mark all edges gray.

- Consider edges in increasing order of weight.
- Add edge to MST (mark black) unless doing so creates a cycle.
- Repeat until $V-1$ edges.



Why does Kruskal's work? Special case of generic MST algorithm.

- Suppose we add edge $e = v \rightarrow w$.
- Side 1 of cut is all vertices connected to v , side 2 is everything else.
- No crossing edge is black (since we don't allow cycles).
- No crossing edge has lower weight (consider in increasing order).

Optimized Kruskal's (Demo)

Lecture 25, CS61B, Spring 2024

Graph Problem Warmup
Minimum Spanning Trees

- Intro
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Prim's Algorithm

- Basic Prim's (Demo)
- Optimized Prim's (Demo)
- Prim's Algorithm Code and Runtime

Kruskal's Algorithm:

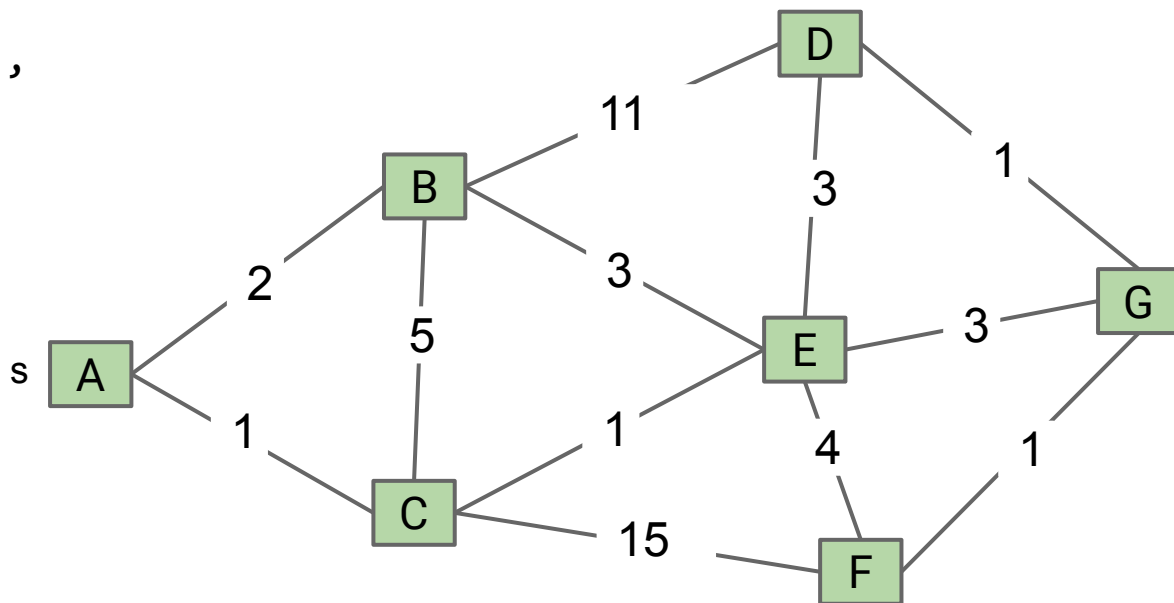
- Basic Kruskal's (Demo)
- **Optimized Kruskal's (Demo)**
- Kruskal's vs. Prim's
- Kruskal's Algorithm Code and Runtime

Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

Fringe: (A-C: 1), (C-E: 1),
(D-G: 1), (F-G: 1),
(A-B: 2), (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)



WQU: []

MST: []

Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

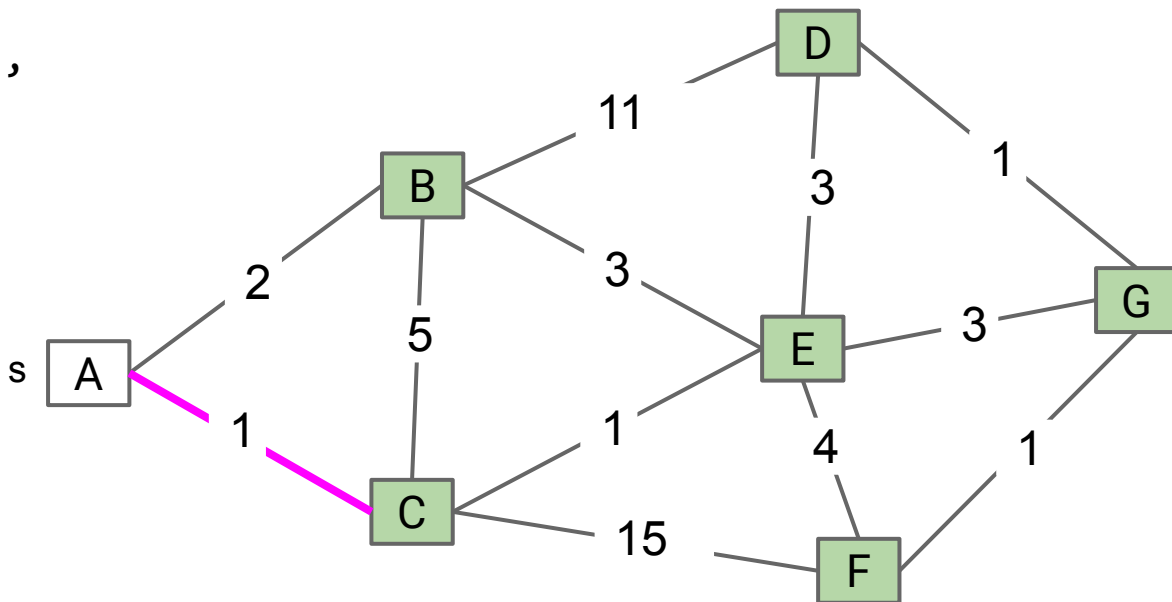
Fringe: ~~(A-C: 1)~~, (C-E: 1),
(D-G: 1), (F-G: 1),
(A-B: 2), (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: A-C

Cycle? isConnected(A, C)

WQU: []

MST: []



Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,

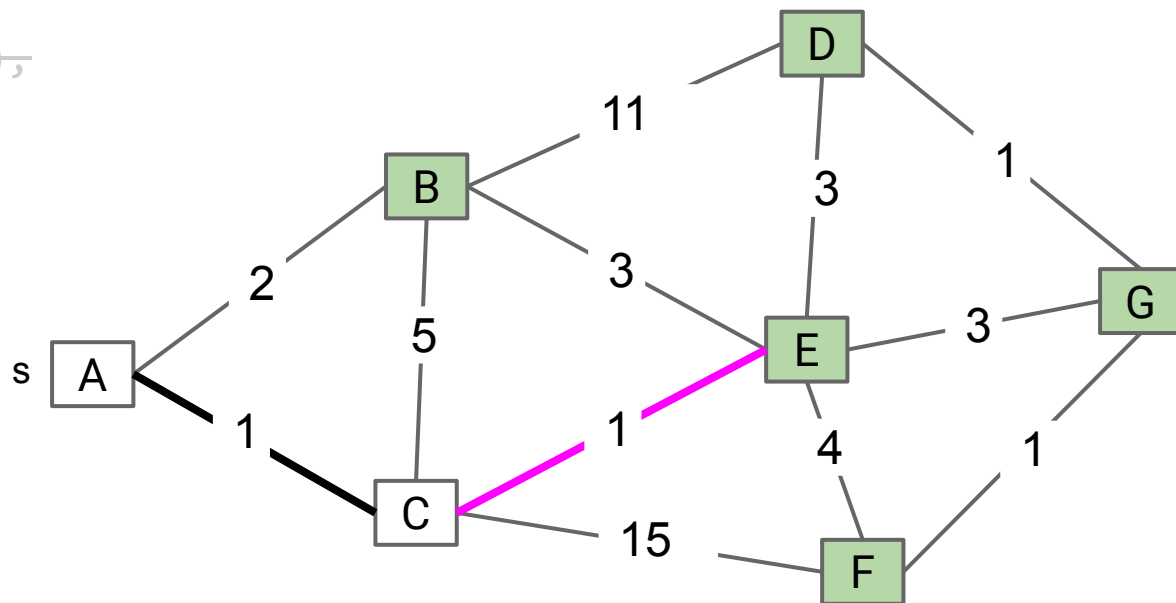
(D-G: 1), (F-G: 1),
(A-B: 2), (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: C-E

Cycle? isConnected(C, E)

WQU: [A-C]

MST: [A-C]



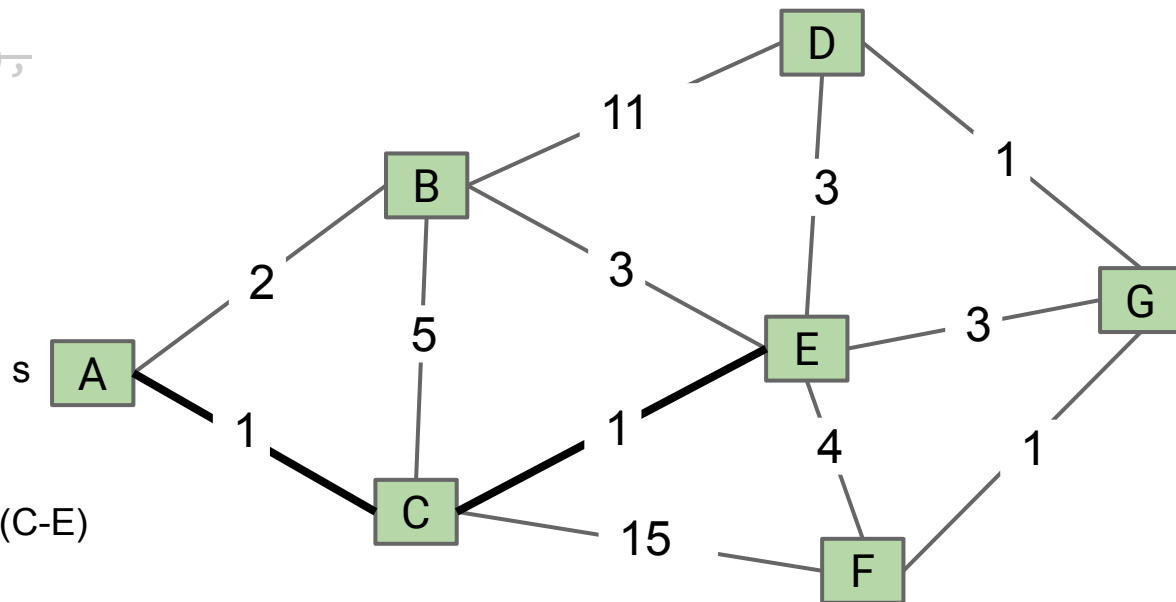
Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,

(D-G: 1), (F-G: 1),
(A-B: 2), (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)



Removed edge: C-E

Cycle? No. union(C, E). add(C-E)

WQU: [A-C-E]

MST: [A-C, C-E]

Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,

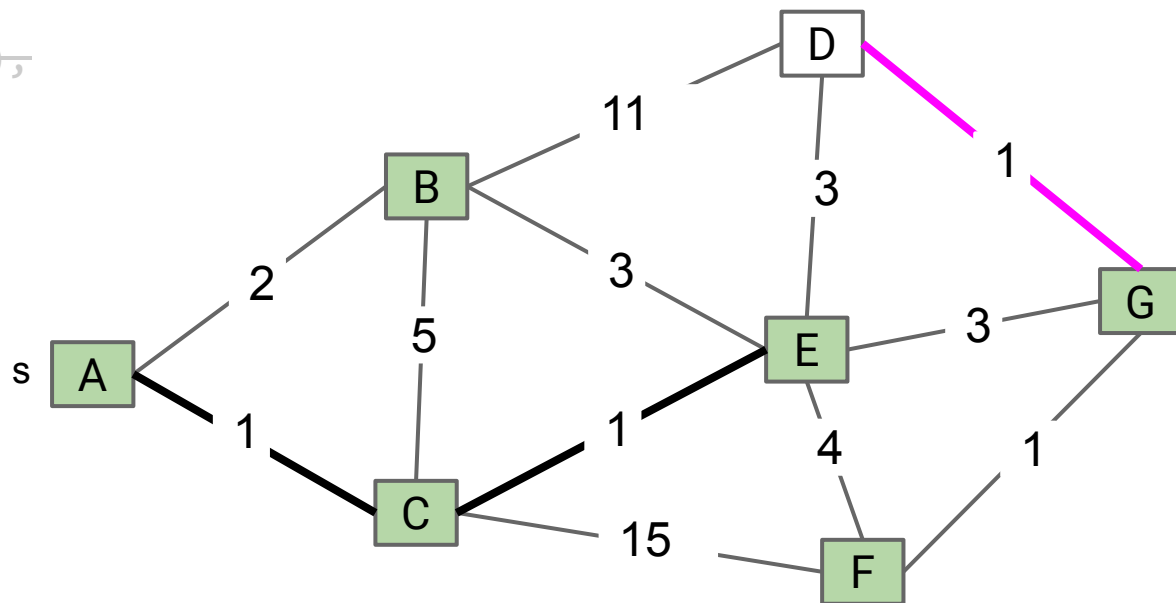
~~(D-G: 1)~~, (F-G: 1),

(A-B: 2), (E-B: 3),

(D-E: 3), (G-E: 3),

(E-F: 4), (B-C: 5),

(B-D: 11), (C-F: 15)



Removed edge: D-G

Cycle? isConnected(D, G)

WQU: [A-C-E]

MST: [A-C, C-E]

Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,

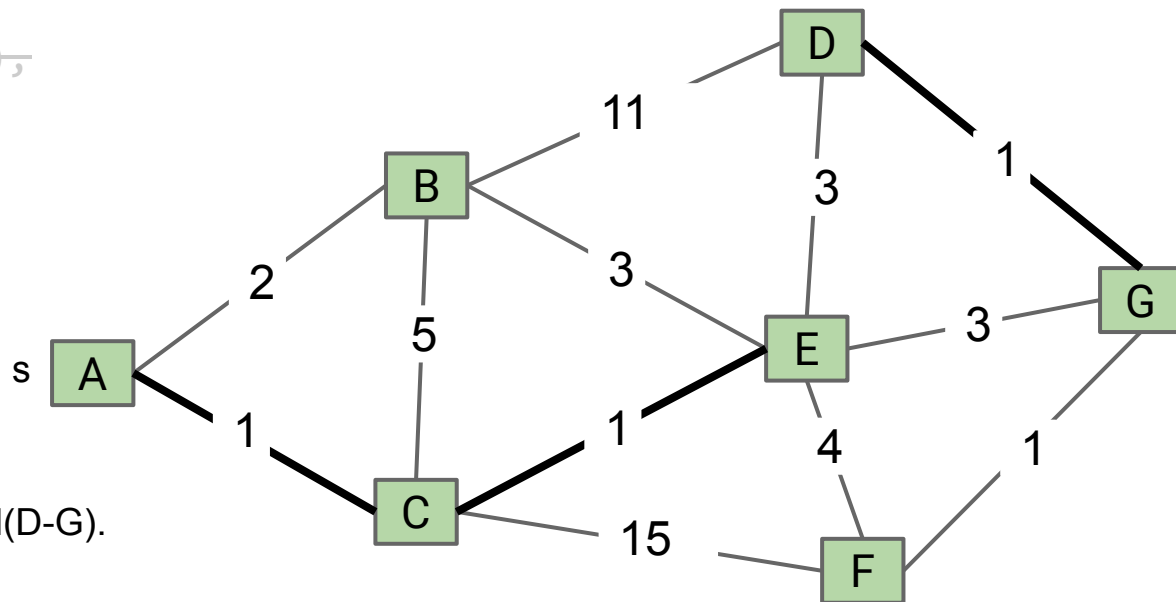
~~(D-G: 1)~~, (F-G: 1),

(A-B: 2), (E-B: 3),

(D-E: 3), (G-E: 3),

(E-F: 4), (B-C: 5),

(B-D: 11), (C-F: 15)



Removed edge: D-G

Cycle? No. union(D, G). add(D-G).

WQU: [A-C-E, **D-G**]

MST: [A-C, C-E, **D-G**]

Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

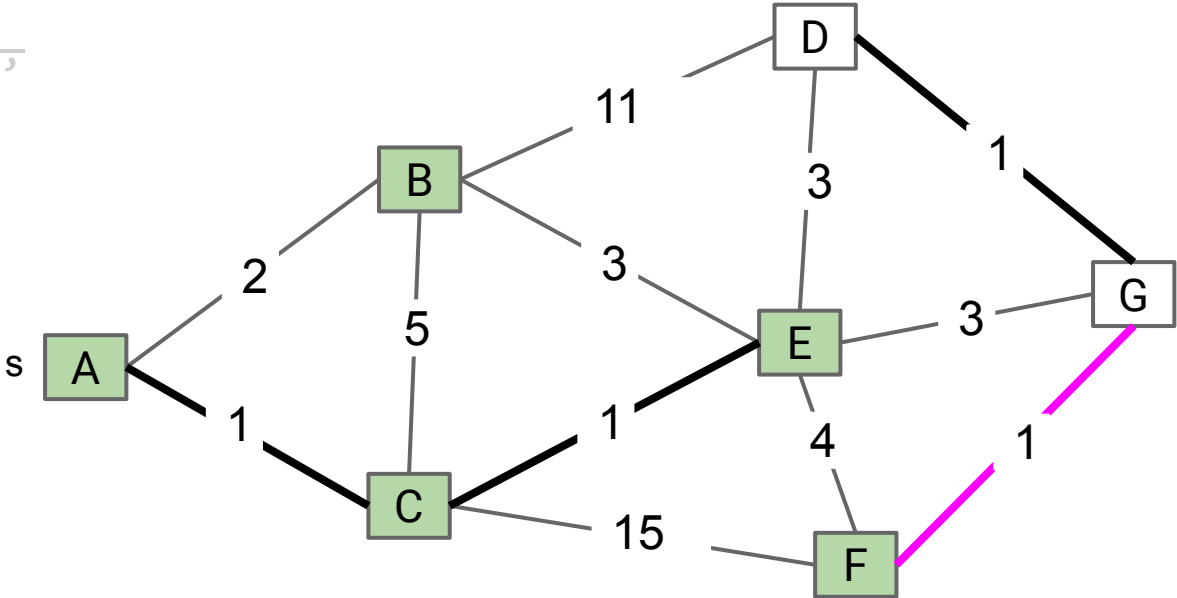
Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,
~~(D-G: 1)~~, ~~(F-G: 1)~~,
(A-B: 2), (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: F-G

Cycle? isConnected(F, G)

WQU: [A-C-E, D-G]

MST: [A-C, C-E, D-G]



Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

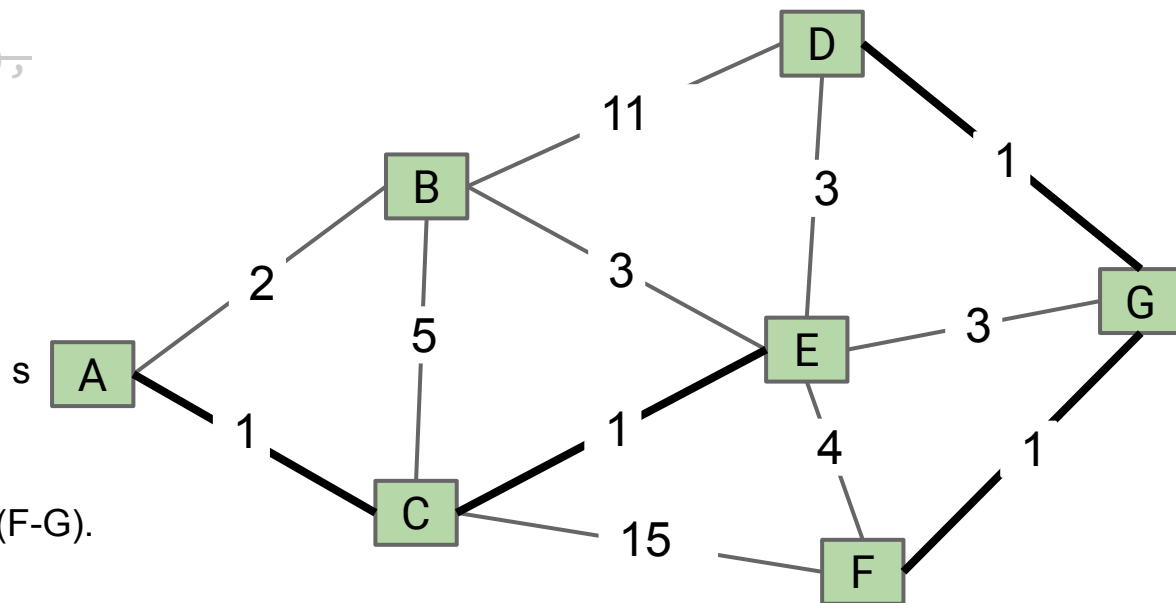
Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,
~~(D-G: 1)~~, ~~(F-G: 1)~~,
(A-B: 2), (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: F-G

Cycle? No. union(F, G). add(F-G).

WQU: [A-C-E, D-G-F]

MST: [A-C, C-E, D-G, F-G]



Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

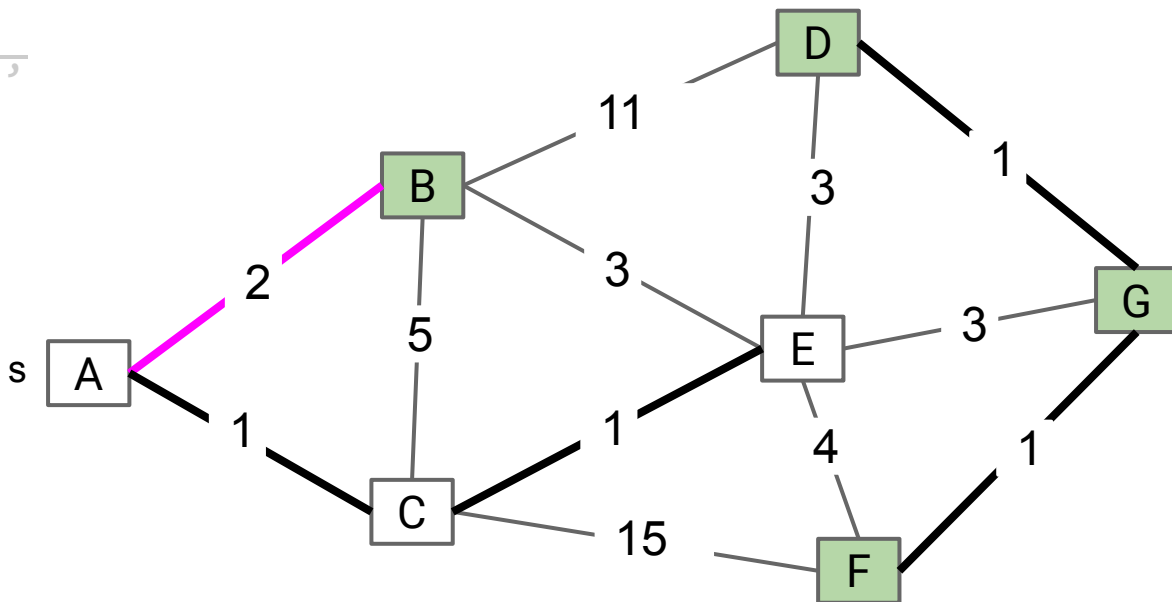
Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,
~~(D-G: 1)~~, ~~(F-G: 1)~~,
~~(A-B: 2)~~, (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: A-B

Cycle? isConnected(A, B)

WQU: [A-C-E, D-G-F]

MST: [A-C, C-E, D-G, F-G]



Kruskal's Demo

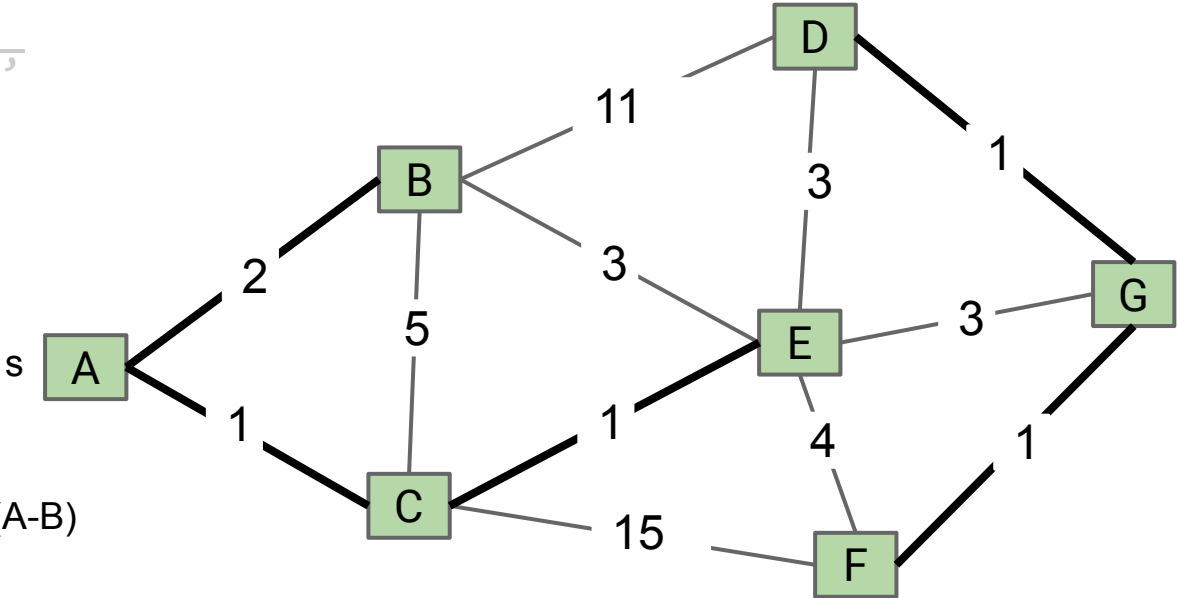
Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,
~~(D-G: 1)~~, ~~(F-G: 1)~~,
~~(A-B: 2)~~, (E-B: 3),
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: A-B

Cycle? No. union(A, B). add(A-B)



WQU: [A-C-E-B, D-G-F]

MST: [A-C, C-E, D-G, F-G, A-B]

Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

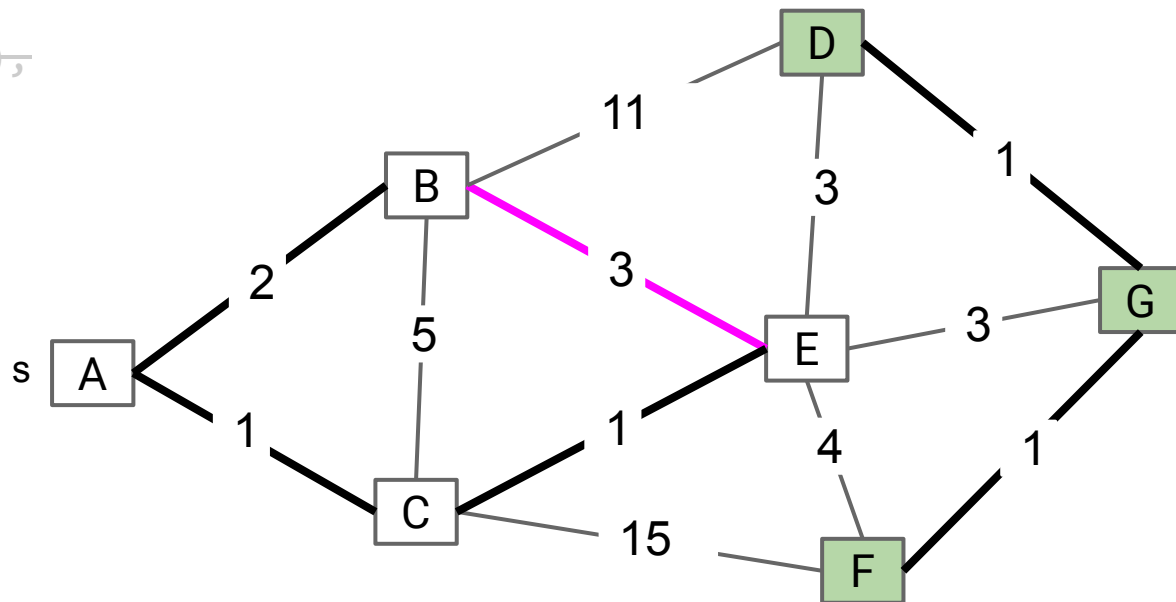
Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,
~~(D-G: 1)~~, ~~(F-G: 1)~~,
~~(A-B: 2)~~, ~~(E-B: 3)~~,
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: E-B

Cycle? isConnected(E, B)

WQU: [A-C-E-B, D-G-F]

MST: [A-C, C-E, D-G, F-G, A-B]



Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

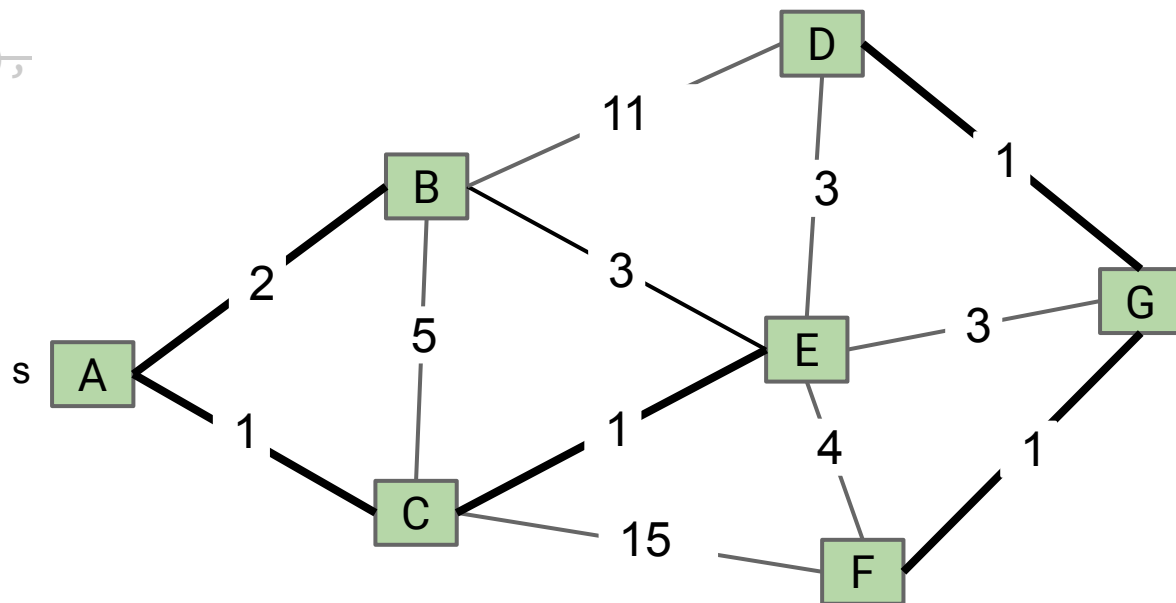
Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,
~~(D-G: 1)~~, ~~(F-G: 1)~~,
~~(A-B: 2)~~, ~~(E-B: 3)~~,
(D-E: 3), (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: E-B

Cycle? Yes. Do nothing.

WQU: [A-C-E-B, D-G-F]

MST: [A-C, C-E, D-G, F-G, A-B]



Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,

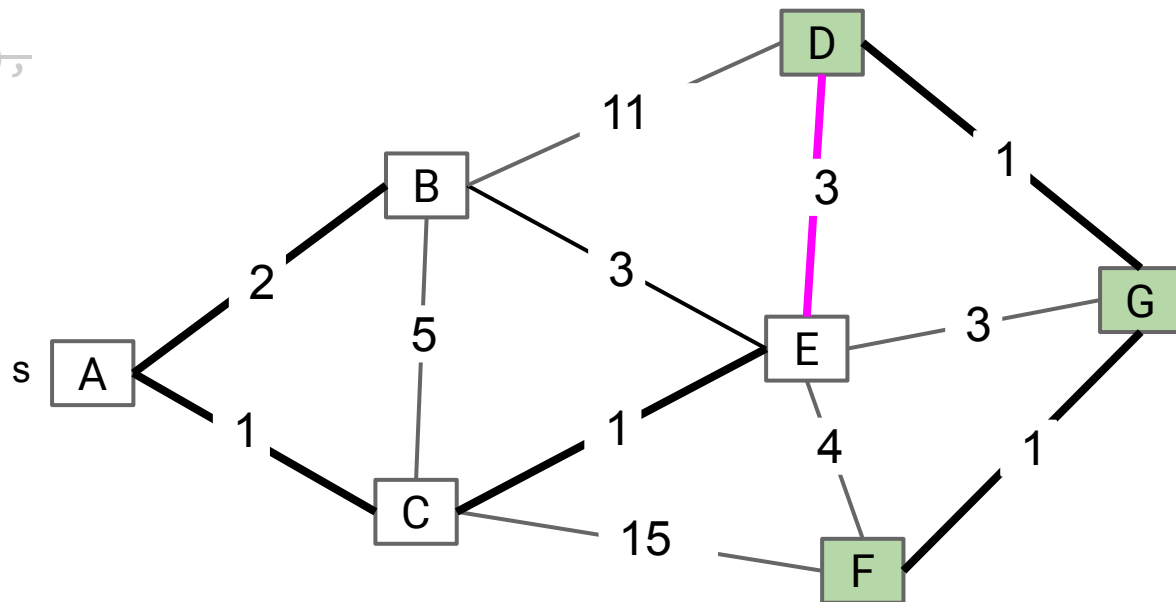
~~(D-G: 1)~~, ~~(F-G: 1)~~,

~~(A-B: 2)~~, ~~(E-B: 3)~~,

~~(D-E: 3)~~, (G-E: 3),

(E-F: 4), (B-C: 5),

(B-D: 11), (C-F: 15)



Removed edge: D-E

Cycle? isConnected(D, E)

WQU: [A-C-E-B, D-G-F]

MST: [A-C, C-E, D-G, F-G, A-B]

Kruskal's Demo

Insert all edges into PQ.

Repeat: Remove smallest weight edge. Add to MST if no cycle created.

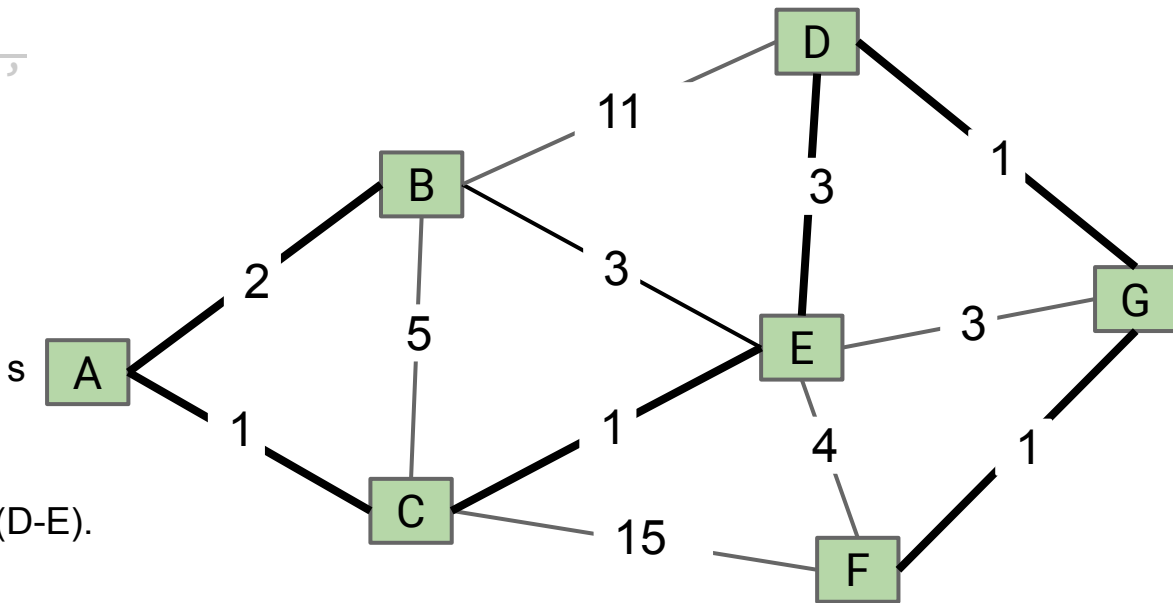
Fringe: ~~(A-C: 1)~~, ~~(C-E: 1)~~,
~~(D-G: 1)~~, ~~(F-G: 1)~~,
~~(A-B: 2)~~, ~~(E-B: 3)~~,
~~(D-E: 3)~~, (G-E: 3),
(E-F: 4), (B-C: 5),
(B-D: 11), (C-F: 15)

Removed edge: D-E

Cycle? No. union(D, E). add(D-E).

WQU: [A-C-E-B-D-G-F]

MST: [A-C, C-E, D-G, F-G, A-B, D-E]



V-1 edges, so done.

Kruskal's vs. Prim's

Lecture 25, CS61B, Spring 2024

Graph Problem Warmup Minimum Spanning Trees

- Intro
- The Cut Property

Prim's Algorithm

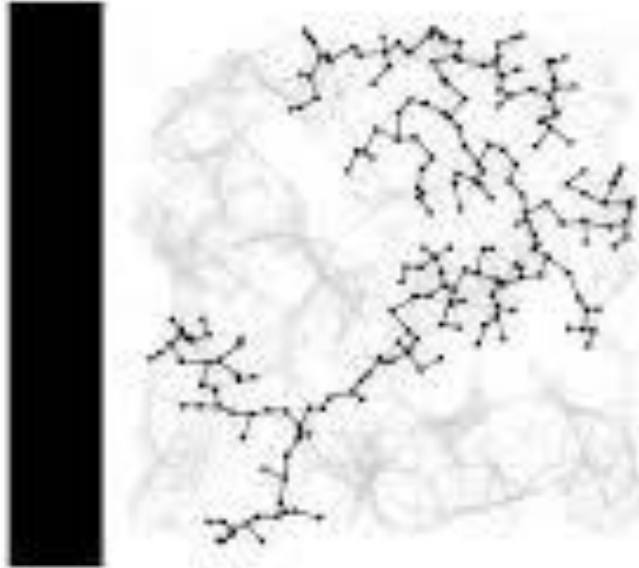
- Basic Prim's (Demo)
- Optimized Prim's (Demo)
- Prim's Algorithm Code and Runtime

Kruskal's Algorithm:

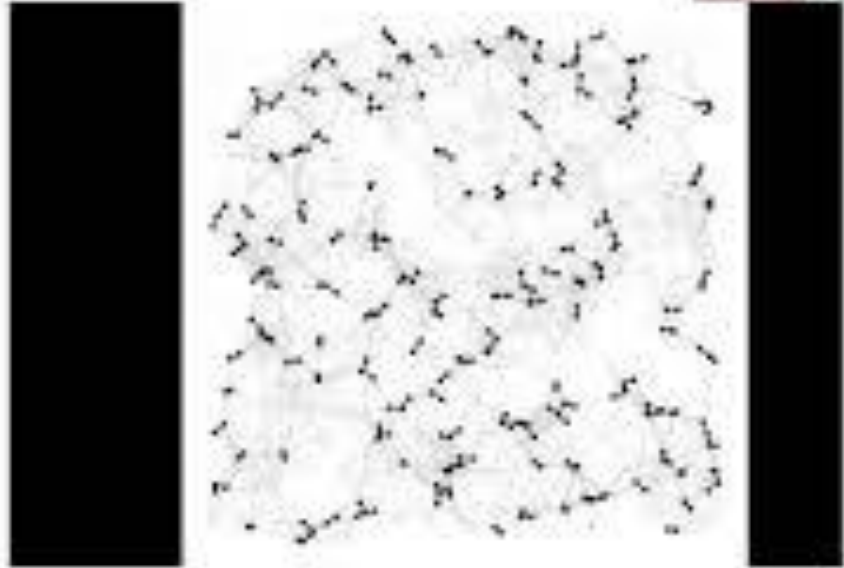
- Basic Kruskal's (Demo)
- Optimized Kruskal's (Demo)
- **Kruskal's vs. Prim's**
- Kruskal's Algorithm Code and Runtime

Prim's vs. Kruskal's (visual)

Prim's vs. Kruskal's



Prim's Algorithm



Kruskal's Algorithm

Demos courtesy of Kevin Wayne, Princeton University



Kruskal's Algorithm Code and Runtime

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Kruskal's Algorithm:

- Basic Kruskal's (Demo)
- Optimized Kruskal's (Demo)
- Kruskal's vs. Prim's
- **Kruskal's Algorithm Code and Runtime**

Kruskal's Implementation (Pseudocode)

```
public class KruskalMST {
    private List<Edge> mst = new ArrayList<Edge>();

    public KruskalMST(EdgeWeightedGraph G) {
        MinPQ<Edge> pq = new MinPQ<Edge>();
        for (Edge e : G.edges()) {
            pq.insert(e);
        }
        WeightedQuickUnionPC uf =
            new WeightedQuickUnionPC(G.V());
        while (!pq.isEmpty() && mst.size() < G.V() - 1) {
            Edge e = pq.delMin();
            int v = e.from();
            int w = e.to();
            if (!uf.connected(v, w)) {
                uf.union(v, w);
                mst.add(e);
            }
        }
    }
}
```

What is the runtime of Kruskal's algorithm?

- Assume all PQ operations take $O(\log(V))$ time.
- Assume all WQU operations take $O(\log^* V)$ time.
- Give your answer in Big O notation.

Kruskal's algorithm on previous slide is $O(E \log E)$.

Fun fact: In HeapSort lecture, we will discuss how do this step in $O(E)$ time using "bottom-up heapification".

| Operation | Number of Times | Time per Operation | Total Time |
|----------------|-----------------|--------------------|-----------------|
| Insert | E | $O(\log E)$ | $O(E \log E)$ |
| Delete minimum | $O(E)$ | $O(\log E)$ | $O(E \log E)$ |
| union | $O(V)$ | $O(\log^* V)$ | $O(V \log^* V)$ |
| isConnected | $O(E)$ | $O(\log^* V)$ | $O(E \log^* V)$ |

Note 1: If we use a pre-sorted list of edges (instead of a PQ), then we can simply iterate through the list in $O(E)$ time, so overall runtime is $O(E + V \log^* V + E \log^* V) = O(E \log^* V)$.

Note 2: $E < V^2$, so $\log E < \log V^2 = 2 \log V$, so $O(E \log E) = O(E \log V)$. So while Kruskal's algorithm will be slower than Prim's algorithm for a worst-case unsorted set of edges, it won't be asymptotically slower.

Shortest Paths and MST Algorithms Summary

| Problem | Algorithm | Runtime (if $E > V$) | Notes |
|----------------|---------------------------------|-----------------------|----------------------------------|
| Shortest Paths | Dijkstra's | $O(E \log V)$ | Fails for negative weight edges. |
| MST | Prim's | $O(E \log V)$ | Analogous to Dijkstra's. |
| MST | Kruskal's | $O(E \log E)$ | Uses WQUPC. |
| MST | Kruskal's with pre-sorted edges | $O(E \log^* V)$ | Uses WQUPC. |

Question: Can we do better than $O(E \log V)$? See bonus slides.

These slides are covered in the [web videos](#), but we won't cover them live.

Extra: MST Algorithm History

Lecture 25, CS61B, Spring 2024

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- Optimized Kruskal's (Demo)
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- Kruskal's Algorithm Code and Runtime

170 Spoiler: State of the Art Compare-Based MST Algorithms

| year | worst case | discovered by |
|------|----------------------------------|----------------------------|
| 1975 | $E \log \log V$ | Yao |
| 1984 | $E \log^* V$ | Fredman-Tarjan |
| 1986 | $E \log (\log^* V)$ | Gabow-Galil-Spencer-Tarjan |
| 1997 | $E \alpha(V) \log \alpha(V)$ | Chazelle |
| 2000 | $E \alpha(V)$ | Chazelle |
| 2002 | optimal (link) | Pettie-Ramachandran |
| ??? | $E ???$ | ??? |

(Slide Courtesy of Kevin Wayne, Princeton University)